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IN ADDITION TO THOSE

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EDITED BY

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$$\frac{(a+b)\cos\theta-(a-b)\cos\phi}{a\cos\alpha} = \frac{(a+b)\sin\theta-(b-a)\sin\phi}{b\sin\alpha} = \frac{2ab}{a^2\cos^2\alpha+b^2\sin^2\alpha},$$

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13992. (R. Tucker, M.A.)—Points D, E, F; D′, E′, F′ are taken on the sides of a triangle ABC, and perpendiculars are let fall from them on the sides. The feet of the respective pairs of perpendiculars are joined, thus forming six triangles. If these triangles are all equal, then the points D, E, F; D′, E′, F′ lie on a circle concentric with the circle ABC 85

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13993. (C. E. Hillyer, M.A.)—AD, BE, CF are the altitudes, H the orthocentre, and I the incentre of a triangle ABC. Prove that the three chords of intersection of the circles AID, BIE, CIF with the incircle meet in the mid-point of IH 73

13999. (R. Knowles, B.A.)—Prove that (1) the $(r+1)$th term in the development of

$$\left\{1+\frac{n(n-1)}{1.2}x\ldots+\frac{n(n-1)\ldots(n-2r+1)}{2r!}x^r\right\}+(1-x)^n$$

$$=\frac{(2r+1)(2r+2)\ldots(2r+n-1)}{n-1!}x^r;$$

e.g., if $n=3$, the general term is $(r+1)(2r+1)x^r$ and the quotient of

$$\frac{1+3x}{(1-x)^3}=1+6x+15x^2+\ldots,$$

putting $r=1, 2, 3$ respectively.

(2) $$\frac{(2r-1)(2r)(2r+1)\ldots(2r+n-3)}{n-1!}$$

is the general term the integral of which in finite differences is the sum of the first r coefficients in any development; if $n=4$, this sum

$$=r(r+1)(2r^2+2r-1)+6 \ldots\ldots 83$$

14004. (Robert W. D. Christie.)—Let N, an integer, consist of p units when p is a prime >3. Prove (1) N is of form $pq+1$; (2) every factor of N is of like form; (3) the sum of the digits of one factor $=pr$.

Ex. gr. $$11111111111=21649\times513239=(px+1)(py+1)$$

and $$2+1+6+4+9=11\times2.$$

Show incidentally that the factors of q in (1) can be written down at sight 75

14016. (Professor Umes Chandra Ghosh.)—Prove that the n-ple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\ldots e^{-(u+v+w+\ldots)}u^{(1-n)/n}v^{(2-n)/n}w^{(3-n)/n}\ldots du\,dv\,dw=\sqrt{\frac{(2\pi)^{n-1}}{n}}$$

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14026. (R. F. Davis, M.A.)—If a, b, c, d be the sides of a cyclic quadrilateral, and $2s=a+b+c+d$, the angle ϕ between its diagonals is given by the equation $\tan\frac{1}{2}\phi=\sqrt{\left\{\frac{(s-a)(s-c)}{(s-b)(s-d)}\right\}}$ 87

14034. (G. W. Preston, B.A.)—O, P are two fixed points in a given straight line, and PB, PC are two other fixed straight lines equally inclined to the given straight line and on the same side of it. If a circle be described, with centre O and any radius, so as to cut PB, PC in B and C, show, geometrically, that the difference of the projections of PB, PC on the given straight line is a constant. [Suggested by a problem in Loney's *Statics*] 28

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14036. (R. W. D. Christie.) — Explain the following instance of curious figuring:— $151 \cdot 10^7 - 1 = 109$ (M).
M is always the series 1, 1, 2, 3, 5, 8, 13, ..., placed in reversed order, *i.e.*, $= 109\,(13853211)$.
Similarly with other values not accidental 71

14052 (B. N. Cama, M.A.) — Show that the sum of all numbers greater than 10^{n-1} which can be formed with the n digits $a_1, a_2, a_3, \ldots, a_n$ (no digit being repeated in the same number) is $\frac{1}{9}S\,(10^n - 1)\,(n-1)!$, S representing the sum of the n digits. Find the corresponding sum when one of the digits (a_n, say) vanishes. 126

14068. (Professor H. Langhorne Orchard, M.A., B.Sc.)—In each of the four triangles formed by intersecting diagonals of a square is inscribed the maximum square. Each of these squares is similarly divided into triangles, in which maximum squares are inscribed; and so on *ad infinitum*. Find the ratio of the areas of all the squares to the areas of all the triangles containing them 51

14076. (Professor Thomas Savage.) — Show that, n being greater than 2, and s (an integer) a perfect nth power,

$$^nCx^{n-1} + {}^nC_2 s x^{n-2} + {}^nC_3 s^2 x^{n-3} + \ldots + {}^nC_{n-1} s^{n-2} x + s^{n-1}$$

cannot be a perfect nth power, when x is any positive number (whole or fractional). Also ascertain if the theorem be true for negative values (integral or fractional) of x 52

14078. (Professor Umes Chandra Ghosh.)—Show that

$$\Gamma\left\{\tfrac{1}{2}(2m+1)\right\} = (2m-1)\,(2m-3)\,(2m-5) \ldots 5 \cdot 3 \cdot 1 \cdot (\sqrt{\pi})/2^m,$$

where m is any integer. Hence show that

$$\Gamma(\tfrac{1}{2}) + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} + \frac{\Gamma(\frac{5}{2})}{\Gamma(3)} + \frac{\Gamma(\frac{7}{2})}{\Gamma(4)} + \&c.\ \text{to}\ \infty = 0 \quad [?] \quad \ldots\ldots\ 87$$

4079. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Exhibit the product of the two series

$1 - 8 + 27 - 64 + 125 - 216 + \ldots$ and $1 - 16 + 81 - 256 + 625 - 1296 + \ldots$,

each to n terms, as a function of the roots of a quadratic equation ... 59

14083. (C. E. Hillyer, M.A.)—Show that the collinearity of the mid-points of the diagonals of a complete quadrilateral is an immediate deduction from Menelaus' theorem, viz., that the ratio compounded of the ratios of the segments into which a straight line divides the sides of a triangle is -1, and its converse 91

14084. (G. Hale Puckle, M.A.)—With origin transferred to a real focus and with parallel axes, the equation $\phi(xy) = 0$ becomes (Quest. 13535, No. 437) $\rho^4(ax^2 + 2hxy + by^2) - 2\rho^2\Delta^{\frac{1}{2}}(ax + \beta y) + \Delta = 0$,
and the equation to the directrix becomes

$$\rho^2(ax + \beta y) - \Delta^{\frac{1}{2}} = 0 \ldots\ldots 53$$

14086. (C. N. Cama, M.A.)—Show that the asymptotes of the curve whose r-p equation (the so-called "pedal" equation) is

$$(a - 3p^4)\,r^4 + 4r^3p^3 - 6r^2p^2 + 4rp = 1$$

all touch a circle of radius $(\frac{1}{3}a)^{\frac{1}{2}}$ 54

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14087. (R. F. Davis, M.A.)—Extract from the *Life and Letters of Lewis Carroll*, by S. D. Collingwood. (Fisher Unwin, 1898.)—"Dec. 19th, 1897 (Sunday). Sat up last night till 4 a.m. over a tempting problem sent me from New York—to find three equal rational-sided right-angled triangles. I found *two* whose sides are 20, 21, 29; 12, 35, 37; but could not find *three*."—Prove that the missing triangle was $7\frac{1}{2}$, 55, and $56\frac{1}{2}$, and find general formulæ for such triangles ... 30

14088. (Rev. T. Roach, M.A.)—OP and OQ are tangents to a parabola: prove $OP^2 : OQ^2 = SP : SQ$ 99

14089. (D. Edwardes, B.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \sin x \log \sin x \log \cos x \, dx = 2 - \log 2 - \tfrac{1}{8}\pi^2 \quad \ldots\ldots 49$$

14090. (G. H. Hardy, B.A.)—Evaluate

$$\int_0^\infty \tan^{-} (p \operatorname{sech} u) \, du \quad \text{and} \quad \int_0^\infty \tan^{-1} \left\{ \tfrac{1}{2} \left(a - \frac{1}{a} \right) \operatorname{sech}^2 u \right\} du \ldots 57$$

14093. (Rev. T. Mitcheson, B.A.)—PQ, a tangent to an ellipse at P, is equal to a diameter parallel to itself: find the locus of Q..... 47

14094. (Robert W. D. Christie.)—In my factorization table (*vide* Vol. LXIX., p. 99) prove the law $a = 10t + u$, $b = Mu + t$, where t signifies the tens digit, u the units, and M the multiplier.

E.g.—Under prime 79 we have

$15 = 10 \times 1 + 5$ and $41 = 8 \times 5 + 1$, $12 = 8 \times 1 + 4$, $17 = 8 \times 2 + 1$, &c.,

where $1501 = 41001 = 120001 = 1700001 = 79$ (M), &c., *ad lib.* 97

14095. (R. Knowles, B.A.)—

$$ax^2 + bxy + cy^2 + ey = 0, \quad a'x^2 + b'xy + c'y^2 + e'y = 0$$

are the equations to two parabolas, touching at P, referred to a tangent and normal at P as axes; if F, F′ are the foci, prove that, if $b : c = b' : c'$, F, F′, P are collinear 46

14096. (G. W. Preston, B.A.)—If a chord of an ellipse make a constant angle with the major axis, show geometrically that the sum of the excentric angles of its extremities is a constant..... 37

14098. (Lt.-Col. Allan Cunningham, R.E.)—Show that the sum of any number (>2) of successive cubes can always be resolved into *three* (algebraic) factors.

E.g.—Resolve the sum of the 8 million cubes from 1,000,001 to 9,000,000 into its prime factors 53

14099. (H. W. Curjel, M.A.)—In a given quadrilateral inscribe a quadrilateral similar to a given quadrilateral 41

14103. (Prof. Crofton, F.R.S.)—If $u = x^2 + ax + b$, a solution of the equation $u\,(d^2y/dx^2) = n\,(n+1)\,y$ is $y = \mathrm{D}^{n-1}u^n$; and another solution is $y = \mathrm{D}^{-n-2}u^{-n-1}$. Also, if $u^2\,(d^4y/dx^4) = n\,(n+1)(n+2)(n+3)\,y$, one solution is $y = \mathrm{D}^{n-1}u^{n+1}$ 38

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14105. (Professor Crofton, F.R.S.)—If $u = x^2 + ax + b$, and we put for shortness the symbol $Du\,D = \omega$, then

$$D^n u^n D^n \equiv \omega\,(\omega - 1\,.\,2)\,(\omega - 2\,.\,3) \ldots \{\omega - (n-1)\,n\} \quad \ldots\ldots\ldots \quad 73$$

14107. (Professor Neuberg.)—Résoudre le système

$$\tan(y+z) = a\tan x, \quad \tan(z+x) = b\tan y, \quad \tan(x+y) = c\tan z \quad 56$$

14110. (Professor Sanjána, M.A.)—In a triangle ABC, K is the symmedian point and O the circumcentre; on KA, KB, KC points A′, B′, C′ are taken such that $KA' : KA = KB' : KB = KC' : KC = e$ (which may be positive or negative and less than unity or greater); through A′, B′, C′ are drawn Y′A′Z, Z′B′X, X′C′Y anti-parallels to BC, CA, AB meeting these sides in X and X′, Y and Y′, Z and Z, respectively. Prove (1) that $Y'Z = Z'X = X'Y = 2(1-e)\,abc/(a^2+b^2+c^2)$; (2) that YZ′ is parallel to BC and $= a\{(2e-1)\,a^2+b^2+c^2\}/(a^2+b^2+c^2)$, with similar results for ZX′ and XY′; (3) that

$$XX' = a\{a^2 + (b^2+c^2)\,(2e-1)\}/(a^2+b^2+c^2),$$

with similar values for YY′ and ZZ′; (4) that the triangles XYZ, X′Y′Z′ are equal to each other and similar to ABC; and that XYZ and ABC have the same positive Brocard point, and X′Y′Z′ and ABC the same negative one; (5) that

$AZ : BX : CY = b/a : c/b : a/c$ and $AY' : CX' : BZ' = c/a : b/c : a/b$;

(6) that the points X, X′, Y, Y′, Z, Z′ are situated on a circle whose centre lies on OK and coincides with the circumcentre of A′B′C′; (7) that the radius of this circle is $R\{e^2 + (1-e)^2\tan^2\omega\}^{\frac{1}{2}}$; (8) that for values 1, ½, 0 of e the circle XX′YY′ZZ′ becomes respectively the circumcircle, the triplicate-ratio circle, the cosine circle; (9) that when

$$e = -\cos A\cos B\cos C,$$

the circle becomes the Taylor circle, and its radius

$$= R(\sin^2 A\sin^2 B\sin^2 C + \cos^2 A\cos^2 B\cos^2 C)^{\frac{1}{2}};$$

(10) that when $e = \sin^2\omega$, the centre of the circle bisects the line joining the Brocard points, and its radius $= R\sin\omega$, which is the least value that the radius can ever have; and (11) that in a right-angled triangle the Taylor and cosine circles coincide. [All properties of Tucker circles can thus be obtained by elementary trigonometry; the result in (11) is inaccurately given in Milne's *Companion* at p. 146.] 34

14113. (D. Biddle.)—From one angle of a parallelogram draw a straight line to each of the opposite sides, such that, with the join of their distal extremities, three equal triangles may be cut off; and prove that the area of the remaining (central) triangle is to one of the former as $\sqrt{5} : 1$ 109

14115. (H. Fortey, M.A.) — A cheese is a cylinder of 2 feet in diameter, and is of variable quality, the part at the axis being worth 1s. 2d. per lb., and the quality deteriorating as the distance from the axis increases, so that at the circumference it is worth only 10d. per lb. An epicure buys a concentric cylindrical piece cut out of the centre and pays 4s. 11d. for it. What is the diameter of this piece? Given that the whole cheese is worth £1. 18s. 3d. 33

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14117. (G. Hale Puckle, M.A.)—By the methods of Quest. 13535, No. 437, show that the equation to a conic, $\phi(xy) = 0$, is equivalent to

$$\rho^2\{a(ax+hy+g)+\beta(hx+by+f)+\Delta^{\frac{1}{2}}\}^2 = (Cx-G-a\Delta^{\frac{1}{2}})^2+(Cy-F-\beta\Delta^{\frac{1}{2}})^2$$

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14119. (Rev. W. Allen Whitworth, M.A.) — If out of m cards numbered 1, 2, 3, ..., m we draw n cards at random, the expectation of the lowest drawn is $(m+1)/(n+1)$ and the expectation of the highest is $n(m+1)/(n+1)$.

[COROLLARY.—If n numbers be named at random, the expectation of the highest is n times the expectation of the lowest.] 75

14122. (The late "Lewis Carroll.") — It is given that (1), if C is true, then, if A is true, B is not true; and (2), if A is true, B is true. Can C be true? What difference in meaning, if any, exists between the following propositions?—(1) A, B, C cannot be all true at once; (2) if C and A are true, B is not true; (3) if C is true, then, if A is true, B is not true 62

14128. (G. W. Preston, B.A.)—Show that

(i.) $\dfrac{\Sigma(y-z)^8}{\Sigma(y-z)^2}-4\Pi(y-z)^2 = \{\Sigma x^2-\Sigma yz\}^3$;

(ii.) $\underset{abcd}{\Sigma}\dfrac{b+c+d}{(b-a)(c-a)(d-a)(x-a)} = \dfrac{a+b+c+d-x}{(x-a)(x-b)(x-c)(x-d)}$.

[See Chrystal's *Algebra*] 41

14133. (Lt.-Col. Allan Cunningham, R.E.) Find a sum of successive odd cubes equal to a square 45

14135. (Professor Thomas Savage.)—For what positive integral value or values of n is $(n+1)^n-n^n$ divisible by $2mn+1$, where m is any positive integer? 94

14137. (Professor Crofton, F.R.S.)—Prove that $(3n)^{4n}-1$ is divisible by $24n+1$, if the latter is a prime number 61

14139. (Professor Neuberg.)—Eliminer x, y, z entre des équations

$$\cos x \sin y \sin z = a, \qquad \cos y \sin z \sin x = b,$$
$$\cos z \sin x \sin y = c, \qquad \cos x \cos y \cos z = d$$ 61

14143. (Professor Umes Chandra Ghosh).—Prove that

$$\int_0^\infty \frac{\cot x - 2^a \cot 2^a . x}{x}\,dx = \tfrac{1}{2}\pi(2^a-1),$$

where a is any positive integer 54

14147. (Rev. W. Allen Whitworth, M.A.) — If three square numbers (not having a common factor) be in arithmetical progression, the middle one divided by 120 will leave a remainder 1 or 25 or 49, and each of the others divided by 240 will leave a remainder 1 or 49. For example: 49, 289, 529; 289, 625, 961; 961, 1681, 2401 98

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14148. (Rev. T. Roach, M.A.)—Sum to infinity the series

$\frac{3}{5}-\frac{9}{11}+\frac{15}{17}-\frac{21}{23}+\ldots$ 51

14149. (B. N. Cama, M.A.)—A map consists of $p+1$ groups of land to be coloured, consisting, respectively, of 1, 2, 3, ..., $(p+1)$ countries meeting at a point, the extreme countries in each group not having a common boundary. Show that the total number of ways of painting the map by $k+1$ colours, when no two adjacent countries in any of the groups are to have the same colour, is $(k^{p+1}-1)(k+1)/(k-1)$. Further, show that, if v_r denote the number of ways of painting a group of u countries in this manner, $v_{r+s}/v_s = k^r$ 103

14150. (R. Knowles, B.A.) — The expression $(1-10x+5x^2)+(1+x)^5$ when expanded becomes the series

$$1-\frac{5\,.\,6}{1\,.\,2}x+\ldots(-1)^r\frac{5\,.\,6\ldots(2r+3)(2r+4)}{1\,.\,2\ldots(2r-1)\,2r}x^r \text{ ad infin.}$$

It is required to deduce the expression from the series 84

14152. (Discipulus.) — An alphabet consists of m consonants and n vowels, m being greater than n. Find the number of ways in which all the letters may be arranged so that each of the vowels shall be between two consonants 86

14153. (Robert W. D. Christie)—Resolve $x^{6m}-x^{3m}+1$ into factors. (This theorem fills the hiatus in the general theorem, Quest. 8098, given in Vol. XLVIII., p. 108) 26

14157. (Rev. T. Mitcheson, B.A.)—A straight line is drawn from O, the intersection of the directrix and the axis of x, meeting an ellipse in P and Q, so that OP = PQ; show that the length of the chord PQ $= \frac{b^2}{4a}\left(\frac{17a^2-9b^2}{a^2-b^2}\right)^{\frac{1}{2}}$, and its inclination to the major axis is $\cos^{-1}\frac{3a}{(17a^2-9b^2)^{\frac{1}{2}}}$ 25

14160. (G. Birtwistle, B.Sc.) — The tangents to any umbilical geodesic on a quadric all intersect the focal hyperbola 46

14164. (Professor Elliott, F.R.S.)—If a, b are positive moduli, and $\alpha-\beta$ lies between 0 and 2π, and if $f(z)$ is a function of the complex variable $z = re^{i\theta}$ which approaches definite limits $f(0)$ and $f(\infty)$ as z comes infinitely small and infinitely great within the sector from $\theta = \beta$ to $\theta = \alpha$, and is holomorphic within and on the bounding lines of the sector except for simple poles at $c_1, c_2, \ldots, c_n$ within the sector, prove the Frullanian formula

$$\int_0^\infty \frac{f(ae^{i\alpha}x)-f(be^{i\beta}x)}{x}dx = \{f(\infty)-f(0)\}\{\log a/b+i(\alpha-\beta)\}-2\pi i\sum_{r=1}^{r=n}\left[(z-c_r)\frac{f(z)}{z}\right]_{z=c_r},$$

the variable x being real 89

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14167. (Professor Neuberg.)—Soient A_1, B_1, C_1, ... les mineurs du déterminant $H = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Démontrer que $(B_1C_2A_3 - B_2C_3A_1) = H(b_1c_2a_3 - b_2c_3a_1)$ 39

14168. (The late Professor Wolstenholme, D.Sc.) — Given a quadric ($u = 0$) and two fixed points O, O', S, S' denote the plane sections of u by the polar planes of O, O'; prove that, if any straight line through O meet the quadric in points P, P', the two cones whose vertices are P, P' and base the plane section S' will intersect in a second plane section whose locus is the quadric ($v = 0$) containing S, S', and such that O is the pole of S', and O' the pole of S. So also, if QQ' be any chord of u passing through O', the locus of the second plane of intersection of the cones whose vertices are Q, Q' and base S is the same quadric 82

14170. (Professor Sanjána, M.A.)—Let Q be the orthocentre of the pedal triangle of ABC; let S, K, T be the circum-, cosine-, and Taylor-centres of ABC. Prove that S, K, T, Q are collinear; that T bisects SQ, and that K divides it in the ratio $1 : 1 + 2\cos A \cos B \cos C$..... 76

14172. (Professor Thomas Savage.)—ABC is an isosceles triangle; DE, a parallel to BC, intersects AB, AC in D, E respectively. If DE, DB, EC are all equal, show that BC is not one of the equal sides of the triangle 59

14175. (R. Tucker, M.A.)—The normal at P, on a parabola, meets the directrix at Q. Show that QS meets (1) the tangent at P, (2) the diameter through P, in cubic curves 89

14176. (R. F. Davis, M.A.)—ABCD is a concyclic quadrilateral, and E, F, G are the intersections (AB, DC), (AD, BC), and (AC, BD) respectively. Show that the focus of the parabola escribed to ABCD is the foot of the perpendicular from G on EF..... 63

14177. (B. N. Cama, M.A.)—Prove that the locus of the intersection of tangents, including a given angle β, to the spiral $r = ae^{\theta \cot \alpha}$ is the copolar spiral $\dfrac{r \sin \psi}{\sin \alpha} = ae^{(\theta + \psi - \alpha)\cot \alpha}$,

where ψ is given by $\cot \psi = \dfrac{k - \cos \beta}{\sin \beta}$

and ak is the vertical distance of a point on the original curve whose vectorial angle is β. Show also that the locus touches the circle round the triangle formed by the tangents and the chord of contact 60

14180. (W. C. Stanham, B.A.)—If O is a fixed point, and PQ is any chord of a conic drawn through one of its foci (S), and if OP, OQ meet the conic again at P' and Q', P'Q' passes through a fixed point (Σ), and, conversely, if P'Q' is any chord through Σ, and if OP', OQ' meet the conic again at P and Q, PQ passes through S..... 61

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14181. (H. MacColl, B.A.)—Let A and B be two events which are not mutually independent. Show that, if the dependence of A upon B is equal to the dependence of B upon A, then the chance of A happening is equal to the chance of B happening. *Definitions.*—The fractional symbol α/β (when α and β are statements) denotes the chance that α is true *on the assumption that* β *is true*; the symbol α/ϵ denotes simply the chance that α is true, *nothing being taken for granted but the data of the problem.* The symbol $\delta\ \alpha/\beta$ denotes $\alpha/\beta - \alpha/\epsilon$, and is called the *dependence of* α *upon* β, or of the event A, asserted by α, upon the event B, asserted by β 79

14182. (Rev. J. Cullen.)—A triangle A′B′C′ is obtained by revolving a given triangle ABC on a point Λ in its plane through an angle θ; A″B″C″ is similarly obtained by revolving ABC on Λ′, the isogonal conjugate of Λ in an opposite direction $(-\theta)$. Prove that (1) the six intersections of the homologous sides of A′B′C′ and A″B″C″ with those of ABC lie on a circle C_θ. (2) As θ varies the locus of the centre of C_θ is the perpendicular to ΛΛ′ through its mid-point. (3) The envelope of C_θ is the inscribed conic whose foci are Λ and Λ′. (4) If Λ lies on the circumcircle of ABC, C_θ becomes a right line whose envelope is a parabola 65

14185. (Rev. T. Mitcheson, B.A.)—If R is the point of intersection of the normals drawn from the ends of semi-conjugate diameters of an ellipse, show that the distance of R from the centre is

$$x_1y_1a^{-3}b^{-3}(a^2-b^2)\left\{b^2(bx_1-ay_1)^2+a^2(bx_1+ay_1)^2\right\}^{\frac{1}{2}},$$

(x_1y_1) being the extremity of one of the diameters 64

14186. (Lt.-Col. Allan Cunningham, R.E.)—Show that the sum of the *cubes* of any number (> 2) of integers in arithmetical progression may always be resolved into *three* factors 51

14189. (V. Daniel.)—XY is the line $x\cos\alpha + y\sin\alpha = p$, referred to rectangular axes through O. If a point move along the zigzag line $OPP_1P_2P_3P_4\ldots$ (which is such that OP is perpendicular to XY, PP_1 to YO, and, generally, P_nP_{n+1} perpendicular to $P_{n-2}P_{n-1}$), it will ultimately arrive at a point S. Similarly, if it move along $OPp_1p_2p_3\ldots$, it will ultimately arrive at a point S′. Prove (1) S and S′ are the foci of an ellipse inscribed in the triangle YOX, and its major axis is

$$p/\sqrt{(1+\cos^2\alpha\sin^2\alpha)}.$$

(2) OX, OY subtend right angles at S and S′ respectively, and hence that three pairs of triangles are similar, namely, OSX, OS′Y; OSY, XS′Y; OS′X, XSY. (3) If SS′ produced intersect the axes in T and T′, $OS^2 \cdot OT' = OS'^2 \cdot OT$. (4) When YOX is isosceles, the ellipse has eccentricity $= 1/\sqrt{5}$ 42

14190. (C. E. Hillyer, M.A.)—If P be a point on the circumcircle of ABC, whose orthocentre is H, and a be the orthocentre of the triangle PBC, b of the triangle PCA, and c of the triangle PAB; prove (1) that B, a, H, C are concyclic; (2) that the Simson-line of P with respect to the triangle ABC is also the Simson-line of a with respect to the triangle HBC, and of b with respect to HCA, and of c with respect to HAB: (3) that this line bisects Aa, Bb, Cc, and PH 91

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14191. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Find the roots of the quadratic equation

$$8x^2 - 2(n^4 + 2n^3 + 3n^2 + 2n)x + n^6 + 3n^5 + 3n^4 + n^3 = 0,$$

and exhibit each as a *series* 72

14192. (Professor Crofton, F.R.S.) — Two points are taken at random in a triangle ABC, the line joining them dividing the triangle into two parts: show that the mean value of the part containing the vertex A is $\frac{1}{2}\frac{4}{7}$ of the triangle. Hence it may be shown that the mean value of the *triangular* part is $\frac{4}{7}$ of the whole 49

14194. (Professor Franz Rogel.)—Prove that

$$S_r = q + \sum_{\kappa=2,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\kappa-1)}}{\kappa!}(2\pi m)^{\kappa-1} \sum_{\nu=1,2,\ldots}^{q} B_\kappa(\nu)\,\nu^{r-\kappa} \quad \ldots\ldots (1),$$

where $S_r = \Sigma t^r$, t divisor of $m \leqq q$, and B_κ denotes the "Bernoullian function" of the κth order.

$$E\left(\frac{m}{n}\right) = m + \frac{1}{2\pi}\sum_{\kappa=2,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\kappa-1)}}{\kappa.\kappa!}\left(\frac{2\pi}{n}\right)^{\kappa} B_\kappa(m+1)\,B_\kappa(n) \ldots (2),$$

where $E\left(\frac{m}{n}\right)$ represents the greatest integer $\leqq \frac{m}{n}$. 125

14195. (Professor Neuberg.)—Soit A'B'C' la projection du triangle ABC sur un plan faisant l'angle ϕ avec le plan ABC. Démontrer que les angles des deux triangles vérifient la relation

$$\cos\phi + \sec\phi = \cot A \cot B' + \cot B \cot C' + \cot C \cot A' + \cot A' \cot B + \cot B' \cot C + \cot C \cot A \quad \ldots\ldots 78$$

14196. (Professor Thomas Savage.)—ABC ... WXA is a rectilinear figure inscribed in a circle; the sum of A and any angle other than B or X is equal to two right angles: show that the figure is a quadrilateral 103

14204. (J. Macleod, M.A.)—B, E are the remote points in which the line of centres of two unequal circles which intersect meets the circumferences; A one of the points of intersection of the circles; BF = BA; ED = EA; $\angle FAC = \frac{1}{2}\angle BAE$. The radius AO produced meets the circumference in G; AD produced meets the same circumference in H, and AC meets in K. Prove AKGH a parallelogram..... 44

14207. (B. N. Cama, M.A. In continuation of Quest. 14177.)—In the case when $\beta = \frac{1}{2}\pi$, it is evident that a parabola can be described touching the spiral at the points of contact, and having the chord of contact for a focal chord. Prove that (1) the latus rectum of the parabola is PQ $\sin^2\phi$, PQ being the chord of contact and ϕ the inclination of PQ to the axis of the curve; (2) the angle ϕ is constant for all pairs of orthogonal tangents to the spiral, it being equal in fact to $2(\psi - \alpha)$, where ψ is as defined in Quest. 14177; (3) the locus of the focus of the parabola for varying positions of PQ is a copolar spiral..... 113

14215. (W. C. Stanham, B.A.)—(1) If P is any point on a rectangular hyperbola which passes through four points A, B, C, D, whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4),

$$P(ABCD) = \frac{(x_2-x_1)(x_4-x_3)+(y_2-y_1)(y_4-y_3)}{(x_3-x_2)(x_4-x_1)+(y_3-y_2)(y_4-y_1)} \quad \ldots\ldots 52$$

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14228. (Professor A. Droz-Farny.)—Un triangle ABC est inscrit dans une conique. Les deux bissectrices de l'angle A rencontrent pour la seconde fois en a et a' la courbe. Démontrer que la droite aa' et ses analogues $\beta\beta'$, $\gamma\gamma'$ se coupent en un même point..... 112

14234. (D. Biddle.)—As the angle A of a variable triangle increases from 0 to π, the sides AB and AC increase in the same proportion with it. Supposing the dimensions at some period of the transformation to be given, and AB to remain fixed in direction, find (1) the maximum height of C above AB, (2) the maximum area attained, (3) the locus of the centroid..... 124

14236. (C. E. Hillyer, M.A.) — O is the circumcentre and I the incentre of a triangle ABC; AO, BO, CO meet the circumcircle again in a, b, c respectively; X, Y, Z are the points of contact of the incircle with the sides, and $X\delta$, $Y\epsilon$, $Z\zeta$ the perpendiculars of the triangle XYZ. Prove that $a\delta$, $b\epsilon$, $c\zeta$ are concurrent in I 57

14237. (R. F. Davis, M.A.)—The tangent at A to the circumcircle of the triangle ABC meets BC in T and is produced to U so that AT = TU. Prove that the polar of A with regard to any circle through B, C passes through U..... 62

14241. (Rev. T. Roach, M.A.) (Suggested by Quest. 13173.)—Six circles are described, each passing through the same excentre of a triangle, and touching one side of the triangle at one of its extremities. (1) Show that the centres of the six circles lie on the radii I_aD, ... on three circles concentric with the excircle. (2) Show that the product of the radii of three alternate circles $= 2R^2r_a$ 107

14242. (C. Joss, M.A.)—Tangents to a circle at the vertices of an inscribed triangle meet the opposite sides produced in collinear points 94

14243. (G. H. Hardy, B.A.)—If p, m are integers, $p > m$, then, if m is odd,

$$\int_0^\pi x \cos px \sin^m x\, dx = (-)^{\frac{1}{2}[m+(-1)^p]} \left\{ \frac{m!\,\pi}{(p^2-1^2)(p^2-3^2)\ldots(p^2-m^2)} \right\};$$

and, if m is even,

$$\int_0^\pi x \sin px \sin^m x\, dx = (-)^{\frac{1}{2}[m+1+(-1)^p]} \left\{ \frac{m!\,\pi}{p(p^2-2^2)(p^2-4^2)\ldots(p^2-m^2)} \right\}$$

..... 80

14246. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely $(1440^{10}+1)$ 117

14254. (C. E. Bickmore, M.A.)—Is the number 78,875,943,472,201 prime or composite? It is a factor of $(10^{25}+1) \div (10^5+1)$; has no factor between 1 and 100,000; and any factor when divided by 200 leaves one of the two remainders 1 or 51 99

14257. (Professor Cochez.)—Etudier $x^4+y^4+3x^2y+y^2=0$ 95

14259. (Professor Neuberg.)—Eliminer x, y, z entre les équations

$$(x+y)(x+z)=ayz, \quad (y+x)(y+z)=bzx, \quad (z+x)(z+y)=cxy \quad 120$$

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14264. (R. Tucker, M.A.)—P is a point on a parabola, focus S and vertex A. With P as centre and radius PS, a circle is described. The tangent to the parabola cuts this circle in L, M, and AL, AM cut the parabola in Q, R. Find the equations to AL, AM, and to the chord QR 105

14273. (Rev. T. Mitcheson, B.A.) — The angle between a semi-diameter of an ellipse and the tangent at its extremity is a *minimum* when the area of the rectangle contained by the coordinates of the tangential point is a *maximum*, *i.e.*, when $x = a/\sqrt{2}$, $y = b/\sqrt{2}$. 128

14275. (R. J. Dallas.)—Being given four points A, B, C, D, prove that the nine-point circles of the four triangles formed by leaving out one point in turn all pass through one point 95

14276. (R. Knowles, B.A.)—At a point R (eccentric angle θ) of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, chords RP, RQ are drawn at right angles; the normal at R meets PQ in M and the curve in N. Prove (1) that $RM : MN = a^4 \sin^2\theta + b^4 \cos^2\theta : a^2b^2$. (2) If R$(am^3, 2am)$ be on the parabola $y^2 = 4ax$, prove that $RM = m^2MN$ 96

14280. (H. A. Webb.)—Show that a particle dropped from rest at any height will, relatively to an observer on the earth, describe a curve on the surface of a paraboloid of revolution whose axis is parallel to the axis of the earth; it being assumed that the earth rotates on its axis uniformly and exerts a constant attraction on the particle 119

14282. (Professor A. Droz-Farny.)—Dans le triangle ABC de forme donnée, tan A = 1, tan B = 2, tan C = 3; démontrer que la droite d'Euler est parallèle au côté AC 119

14287. (Professor Thomas Savage.)—Find the sums of the angles made with the axis of x by the lines joining the origin to the points of intersection of (1) $x + y = 1$ and $x^3 - x^2y - xy^2 + y^3 = 0$; (2) of $x + y = 1$ and $x^3 - x^2y + xy^2 + y^3 = 0$..... 114

14291. (R. F. Davis, M.A.)—If ABCD be any quadrilateral circumscribed to a circle (centre O), then $AB \cdot AD/OA^2 = CB \cdot CD/OC^2$ ... 96

14306. (Professor G. B. Mathews, F.R.S.)—Prove that, if p, q be odd primes such that p lies between q and $3q$, and if p'/q' be the penultimate convergent in the expansion of p/q as an ordinary continued fraction, then there is a multiple of pq between $(p+q)pq'$ and $(p+q)p'q$. Show, also, that $p+q$ is the least even number $2n$ such that a multiple of pq lies between $2npq'$ and $2np'q$.

Show, further, that, if $2n$ be an assigned even integer, and if odd primes p, q can be found such that $p > \frac{1}{2}n$, $q > \frac{1}{2}n$, and a multiple of pq lie between $2npq'$ and $2np'q$ when $pq' - p'q = \pm 1$, then $2n = p + q$ 108

14314. (R. Tucker, M.A.)—AC, BD are two parallel straight lines which are cut at right angles by OAB, and obliquely by OCD ($\angle COA = \theta$). CP, DQ are parallel to OAB, and DP, CQ perpendicular to OCD. Prove loci of P, Q to be parabolas. Tangents at extremities of their latera recta are perpendicular (or parallel), and the points of intersection of these tangents lie on a hyperbola when BD varies in position..... 114

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14318. (H. MacColl, B.A.)—Two intersecting circles A and B are wholly enclosed in a circle E. Six points are marked in A outside B, three points in B outside A, two points in area common to A and B, and x points in E outside both A and B. Out of the total $x+11$ points in E a point P is taken at random. Let α denote the statement that P will be in A, and β the statement that P will be in B. What must be the value of x (1) so that α shall be independent of β, (2) so that the dependence of α upon β shall be 2/105, (3) so that the dependence of α upon β shall be $-2/95$? [The *dependence of α upon β* means the increase or (when negative) diminution produced in the chance of α by assuming β] ... 117

14319. (Rev. T. Mitcheson, B.A.)—Solve

$$x^3+\tfrac{1}{2}x^2(\alpha-\beta\gamma+\alpha\beta-\gamma+\alpha\beta\gamma^{-1})=\tfrac{1}{2}\alpha\beta\gamma \quad \ldots\ldots\ldots 110$$

14323. (V. Daniel.)—If the images of the vertices A, B, C of a triangle (area $=\Delta$), in the opposite sides, form a triangle of area Δ'. show that $\Delta'/\Delta = 3+8\cos A\cos B\cos C$. Hence show that, when the images are collinear, the triangle ABC has an obtuse angle lying between the limits 120° and $\cos^{-1}(-\sqrt{\frac{2}{3}})$, *i.e.* nearly 127° 46′ 118

14324. (W. S. Cooney.)—P, Q, R are the centres of the insquares of the triangle ABC, to the sides a, b, c respectively. (1) AQ and AR are isogonal lines to b and c, &c. (2) If triangle be right-angled, given P, Q, and centre of inscribed circle, construct triangle. (3) In (2), if P and Q alone be given, area of minimum right-angled triangle equals $36\,(PQ)^2$ 107

14330. (R. Chartres.) — BC is the fixed base of the triangle ABC, G the centroid, and K the symmedian point. Show that the maximum value of the triangle AGK : its mean value :: π : 2 if A describe a semicircle 110

14332. (Professor M. W. Crofton, F.R.S.)—(1) If $4n-1$ is a prime, $1^2 2^2 3^2 \ldots (2n-1)^2-1$ is a multiple of $4n-1$. (2) If $4n+1$ is a prime, $1^2 2^2 3^2 \ldots (2n)^2+1$ is a multiple of $4n+1$. It hence follows that $4n+1$ is the sum of two squares 105

14341. (J. A. Third, M.A.)—K is a conic circumscribed to the triangle of reference and passing through the centroid; K′ is another conic passing through the mid-points of the sides and touching K at the centroid; P is the centre of a conic which touches the sides at points which connect with the opposite vertices through a point on K′; Q is the point of concurrence of lines drawn from the vertices to meet the opposite sides at points where these sides are touched by a conic whose centre lies on K′. Show that PQ touches K and K′ at the centroid 106

14354. (H. W. Curjel, M.A.)—If the ordinate of a point P on an ellipse cuts the auxiliary circle in Q, the distance of the focus S from the tangent at Q = SP 106

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS

14157. (Rev. T. Mitcheson, B.A.)—A straight line is drawn from O, the intersection of the directrix and the axis of x, meeting an ellipse in P and Q, so that OP = PQ; show that the length of the chord PQ $= \frac{b^2}{4a}\left(\frac{17a^2-9b^2}{a^2-b^2}\right)^{\frac{1}{2}}$, and its inclination to the major axis is $\cos^{-1}\frac{3a}{(17a^2-9b^2)^{\frac{1}{2}}}$.

Solution by G. Birtwistle, B.Sc.; F. H. Peachell, B.A.; *and the* Proposer.

Oy is directrix, Cx major axis.

Equation of ellipse referred to Ox and Oy as axes is

$$\frac{1}{a^2}\left(x-\frac{a}{e}\right)^2+\frac{y^2}{b^2}=1$$

or

$$\frac{1}{a^2}\left(r\cos\theta-\frac{a}{e}\right)^2+\frac{r^2\sin^2\theta}{b^2}=1.$$

Therefore
$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)-\frac{2r\cos\theta}{ae}+\frac{b^2}{a^2e^2}=0$$

or
$$Ar^2+2Br+C=0.$$

If r_1 and r_2 are roots of this, $r_1 = OP$, $r_2 = OQ$; therefore $r_2 = 2r_1$;

therefore
$$2r_1^2=\frac{C}{A},\quad 3r_1=\frac{-2B}{A};\qquad \text{therefore } 9AC=8B^2;$$

therefore
$$8\cos^2\theta=9b^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right);$$

therefore
$$8a^2\cos^2\theta=9\left\{b^2\cos^2\theta+a^2(1-\cos^2\theta)\right\};$$

therefore
$$\cos^2\theta=\frac{9a^2}{17a^2-9b^2}.$$

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Also $$PQ = r_1 = \frac{-2B}{3A} = \frac{2\cos\theta}{3ae}\sqrt{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}$$
$$= \frac{b^2}{4a^2e}\sqrt{(17a^2 - 9b^2)} = \frac{b^2}{4a}\sqrt{\left(\frac{17a^2 - 9b^2}{a^2 - b^2}\right)},$$
using the value of $\cos\theta$ just found.

14153. (ROBERT W. D. CHRISTIE) — Resolve $x^{6m} - x^{3m} + 1$ into factors. (This theorem fills the hiatus in the general theorem, Quest. 8098, given in Vol. XLVIII., p. 108.)

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

In all cases $$x^{6m} - x^{3m} + 1 = \frac{x^{9m} + 1}{x^{3m} + 1}.$$

When m is *odd* (but not $= 3\mu$), this is *known* to be further divisible by $(x^9 + 1) \div (x^3 + 1)$; *i.e.*, by $(x^6 - x^3 + 1)$. By actual division it will be found that the quotient consists of a set of pairs of terms $\pm(x^{3\epsilon} + x^{3\epsilon+3})$, which can be easily written down at sight, together with certain central terms. Thus, *commencing from both ends*, the quotient is

$$(x^{6m-6} + x^{6m-9}) - (x^{6m-15} + x^{6m-18}) + (x^{6m-24} + x^{6m-27}) - (x^{6m-33} + x^{6m-36}) + \&c.,$$
$$(1 + x^3) - (x^9 + x^{12}) + (x^{18} + x^{21}) - (x^{27} + x^{30}) + \&c.;$$

and, when $m = 3\mu$, the two series coalesce.

When $m = 3\mu + 1$, the two series stop at a *single* middle term $\pm x^{3(m-1)}$.

When $m = 3\mu + 2$, the two series stop at a *group* of central terms
$$-(x^{3(m-2)} + x^{3(m-1)} + x^{3m}).$$

When m is an *odd* prime, the above is the only algebraic factorization; but, when m is an *odd composite*, there will be other *algebraic* factors depending on the prime factors in m.

When $m = k\mu$, where $k = 2^\kappa$, 3^κ, or 6^κ, and μ is prime to 6, then, writing $x^k = \xi$, $x^{6m} - x^{3m} + 1 = \xi^{6\mu} - \xi^{3\mu} + 1$, which is of the same form as the original, and therefore follows similar rules as to resolvability.

When $m = 1$ or 2^κ, 3^κ, or 6^κ the function is not *generally* resolvable.

The quotient (after removal of the above factor) is further *algebraically resolvable into two factors* in the three following cases:—(1) when $x = 2\xi^2$, $m = 4i + 2$; (2) when $x = 3\xi^2$, $m = 2i + 1$; (3) when $x = 6\xi^2$, $m = 4i + 2$; as both $(x^{9m} + 1)$ and $(x^{3m} + 1)$ then become *Aurifeuillians* of order 2, 3, or 6 in cases (1), (2), (3) respectively. (See the writer's paper, "On Aurifeuillians," in *Proc. Lond. Math. Soc.*, Vol. XXIX.)

Note on Quest. 14153 *by the* PROPOSER.

Besides the usual well known algebraical solutions of $x^{6m} - x^{3m} + 1$, viz., m odd and *prime to* 3, *i.e.*, $m = 6k \pm 1$, and other particular forms,

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a large number of other solutions are obtainable by the doctrine of primitive roots, a few theorems in which I now give.

Let $6m+1$ be a prime and r its primitive root; then

$$r^{2m}-r^m+1 = \mathrm{P}\,(\mathrm{M}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

for, by FERMAT's theorem, $r^{p-1} \equiv 1 \bmod p$, when r is prime to p;

so that $$r^{2m}-r^m+1 \equiv 0 \bmod (6m+1).$$

This congruence can readily be adapted to the required form by making $m = 3\mu$. Some examples are (1) $p = 19$, $m = 3$, $r = 2, 3, 10, 13, 14, 15$, &c.

Then $$2^6-2^3+1 = 3^6-3^3+1 = 10^6-10^3+1$$
$$= 13^6-13^3+1 = \ldots = 19\,(\mathrm{M}).$$

(2) $p = 37$, $m = 6$, $r = 2, 5, 13$, &c.; therefore

$$2^{12}-2^6+1 = 5^{12}-5^6+1 = 13^{12}-13^6+1 = \ldots = 37\,(\mathrm{M}).$$

(3) $p = 73$, $m = 12$, $r = 5$, &c.; therefore $5^{24}-5^{12}+1 = 73\,(\mathrm{M})$.

(4) $p = 109$, $m = 18$, $r = 10$, &c.; therefore $10^{36}-10^{18}+1 = 109\,(\mathrm{M})$, &c., &c.

Another theorem, in some respects more general, is

$r^{\frac{1}{2}(p-1)}+1 = 0 \bmod p$, where r is, as before, any primitive root of p ... (2).

Ex. gr., $p = 19$, $r = 2, 3, 10, 13, 14, 15, 8, 12$, &c., $p = 127$, $r = 7$, &c.; thus $$7^{63}+1 = 7^{42}-7^{21}+1 = 0 \bmod 127.$$

Again, in a former volume (*v.* Quest. 12728, Vol. LXIV.) it was shown that, if $\mathrm{P} = a^2+b^2 = 4m+1$ = a prime and R its primitive root,

then $$\mathrm{R}^m = \pm(ac+bd) \bmod \mathrm{P},$$

and $$\mathrm{R}^{3m} = \mp(ac+bd) \bmod \mathrm{P}, \text{ where } ad-bc = \pm 1 \ldots\ldots\ldots (3),$$

and its cognate theorem $\mathrm{P} = \mathrm{R}a^2+b^2 = 4m+3$.

$$\mathrm{R}^{m+1} = \mp(\mathrm{R}ac+bd) \bmod \mathrm{P},$$
$$\mathrm{R}^{3m+2} = \pm(\mathrm{R}ac+bd) \bmod \mathrm{P}, \text{ where } ad-bc = \pm 1 \ldots\ldots\ldots (4),$$

and others might be given.

Again, we can obtain values by the factorization of $x^2 \pm x+1$, which may be effected in many ways.

Let $x = (y^2+y+1)\,k+y$; then the factors are

$$(y^2+y+1)(k^2y^2+k^2y+2ky+k^2+k+1) \ldots\ldots\ldots\ldots (5).$$

Again, $x = 2k^2+k+1$; then the factors are

$$(k^2+k+1)(4k^2+3) \ldots\ldots\ldots\ldots (6).$$

Also $x = k\,(k+1)(k^2+1)$; then the factors are

$$(k^2+k+1)(k^6+k^5+k^4+2k^3+k^2+1) \ldots\ldots\ldots\ldots (7),$$

and $x = k_\sigma^4 = k^4+k^3+k^2+k+1$; then the factors are

$$(k^2-k+1)(k^6+3k^5+5k^4+6k^3+7k^2+6k+3) \ldots\ldots\ldots\ldots (8),$$

&c., &c.

To change x^2+x+1 into x^2-x+1, merely let $x = x-1$.

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To adapt this latter form to our purpose, let us take (5) *supra*. $y = 1$ gives $x = 3k + 1 = z^3$ say ; then we have

$$z^6 + z^3 + 1 = (y^2 + y + 1)(k^2y^2 + k^2y + 2ky + k^2 + k + 1),$$

and its cognate form

$$z^6 - z^3 + 1 = (y^2 - y + 1)(k^2y^2 - k^2y + 2ky - k^2 - k + 1).$$

Examples are $y = 2$, $k = 2$; then

$$2^6 - 2^3 + 1 = 3 \times (16 - 8 + 8 + 4 - 2 + 1) = 3 \times 19.$$

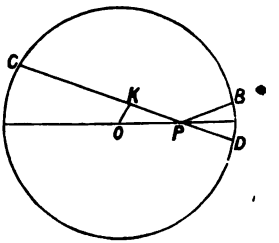
$y = 5$, $k = 41$; then

$$5^6 - 5^3 + 1 = 3 \times (6724 - 3362 + 164 + 1681 - 40 + 1) = 3 \times 5167,$$

and, by ordinary manipulation, factors may be got from (6), (7), (8), &c.

14034. (G. W. Preston, B.A.)—O, P are two fixed points in a given straight line, and PB, PC are two other fixed straight lines equally inclined to the given straight line and on the same side of it. If a circle be described, with centre O and any radius, so as to cut PB, PC in B and C, show, geometrically, that the difference of the projections of PB, PC on the given straight line is a constant. [Suggested by a problem in Loney's *Statics*.]

Solution by Lionel E. Reay, B.A. ; H. W. Curjel, M.A. ; *and the* Proposer.



OK is perpendicular to PC. CP is produced to D. PD = PB and their projections on OP are equal.

$$CP - PB = 2PK = \text{const.},$$

since OP and $\angle$ OPC are const. ; therefore the difference of the projections of CP, PB on OP = twice the projection of PK = constant.

14016. (Professor Umes Chandra Ghosh.) — Prove that the n-ple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \dots e^{-(u+v+w+\dots)}\, u^{(1-n)/n}\, v^{(2-n)/n}\, w^{(3-n)/n} \dots du\, dv\, dw = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

Solution by H. W. Curjel, M.A.

The given integral evidently

$$= \Gamma(1/n)\, \Gamma(2/n) \dots \Gamma(n/n)$$

$$= \sqrt{\{\Gamma(1/n)\, \Gamma(1-1/n)\, \Gamma(2/n)\, \Gamma(1-2/n) \dots \Gamma[(n-1)/n]\, \Gamma[1-(n-1)/n]\}}$$

$$= \sqrt{\left\{\frac{\pi^{n-1}}{\sin \frac{1}{n}\pi \sin 2\pi/n \dots \sin[(n-1)\pi/n]}\right\}} = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

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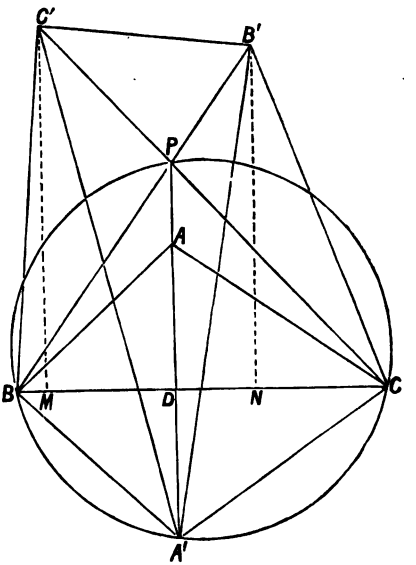
13969. (C. WAFELBAKKER.) — The triangle ABC is supposed to revolve round the axis BC, till the vertex A, having described a semi-circle, has reached the position A′. In the same manner the points B′ and C′ are derived from B and C. It is asked how to construct ABC, being given the points A′, B′, C′.

Remarks by D. BIDDLE.

The following facts, relating to this Question, are noteworthy:—(1) BB′, CC′ intersect in P, the orthocentre of ABC, at an angle supplementary to A, and A′, B, P, C are concyclic.



(2) $A'BC' = 3B$,
and $A'CB' = 3C$;
also $BC' = CB' = a$.

(3) $(A'B')^2 - c^2 : (A'C')^2 - b^2$
$= \sin 2C : \sin 2B$
$= B'N : C'M$,
whence

$$\{(A'B')^2 - c^2\}\, b \cos B = \{(A'C')^2 - b^2\}\, c \cos C;$$

also
$$\frac{ab\{(A'B')^2 - c^2\}}{c(a^2 + b^2 - c^2)} = \frac{ac\{(A'C')^2 - b^2\}}{b(a^2 + c^2 - b^2)} = \frac{bc\{(B'C')^2 - a^2\}}{a(b^2 + c^2 - a^2)}.$$

And, of course, in the case of a right-angled original triangle, one of the unknown sides is equal to one of the known values.

Let $\theta = AA'B'$ and $A'A = 2x \cos\theta$,
whilst the sides and angles of $A'B'C'$ are a', b', c', α, β, γ. Then (4)

$$(c' - 2x)^2 \cos^2\theta + c'^2 \sin^2\theta = c^2, \quad \{b' \cos(\alpha - \theta) - 2x \cos\theta\}^2 + b'^2 \sin^2(\alpha - \theta) = b^2,$$

and
$$c^2 : b^2 = \sin^2 C : \sin^2 B = BN : CM.$$

Now
$$MN = c' \sin\theta + b' \sin(\alpha - \theta)$$

and
$$BM : CN = 1 - 2\sin^2 B : 1 - 2\sin^2 C.$$

(5) Four equal circles can be drawn, namely, A′CPB, B′PAC, C′PAB, and the fourth, with centre P, passing through the centres of the other three, the joins forming $\Delta = ABC$. Therefore a fifth circle, equal to the others, is the circumcircle of ABC.

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[This Question, having long been published on the Continent without eliciting any satisfactory solution, was, in hopes of better things, sent to the *Educational Times*, where, however, it seems likely to meet with the same fate. No solution whatever has come to hand.]

14087. (R. F. Davis, M.A.)—Extract from the *Life and Letters of Lewis Carroll*, by S. D. Collingwood. (Fisher Unwin, 1898.)—"Dec. 19th, 1897 (Sunday). Sat up last night till 4 a.m. over a tempting problem sent me from New York—to find three equal rational-sided right-angled triangles. I found *two* whose sides are 20, 21, 29; 12, 35, 37; but could not find *three*."—Prove that the missing triangle was $7\frac{1}{2}$, 56, and $56\frac{1}{2}$, and find general formulæ for such triangles.

Solution by C. E. Hillyer, M.A.

If a, b, c be the sides of a right-angled triangle,

$$c^2 = a^2 + b^2 \quad \text{or} \quad a^2 = c^2 - b^2 = (c+b)(c-b).$$

Suppose $$a = m/n\,(c-b) = n/m\,(c+b)\,;$$

then $$an + bm = cm \quad \text{and} \quad am - bn = cn\,;$$

therefore $$a = \frac{2mn}{m^2+n^2}c \quad \text{and} \quad b = \frac{m^2-n^2}{m^2+n^2}c.$$

The formulæ, then, for all integral-sided right-angled triangles are

$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2,$$

where m and n stand for any two numbers of which $m > n$, and the area of the triangle $= \frac{1}{2}ab = mn\,(m^2-n^2)$ or $mn\,(m+n)(m-n)$.

We have to find as many as possible different pairs of integers m, n, making the product $mn\,(m+n)(m-n)$ the same.

Now, if we find equal triangles by giving *fractional* values to m and n, we shall only have to multiply by the least common denominator to deduce as many equal triangles with *integral* sides; we may therefore remove the restriction that m and n are to be integers, so that we may suppose either that $m = 1$ and $n =$ some positive number less than 1, or that $n = 1$ and $m =$ some positive number greater than 1.

In the first case, suppose

$$n\,(1+n)\,(1-n) = n'\,(1+n')\,(1-n')\,;$$

then $$n - n^3 = n' - n'^3 \quad \text{or} \quad n - n' = n^3 - n'^3,$$

and, since $n \neq n'$, $$n^2 + nn' + n'^2 = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

In the second case, suppose

$$n'\,(1+n')(1-n') = m'\,(m'+1)(m'-1)\,;$$

then, n' not being equal to m', we have, similarly,

$$n'^2 - n'm' + m'^2 = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

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From (1), $n = \frac{1}{2}\{-n' \pm \sqrt{(4-3n'^2)}\}$,

and we must have $4-3n'^2 = x^2$ or $3n'^2 = (2+x)(2-x)$.

Suppose, then, $3n' = p/q\,(2+x)$, $n' = q/p\,(2-x)$,

whence $x = \dfrac{2\,(3q^2-p^2)}{3q^2+p^2}$, $n' = \dfrac{4pq}{3q^2+p^2}$,

and $n = \frac{1}{2}(-n' \pm x) = \dfrac{3q^2-2pq-p^2}{3q^2+p^2}$ or $\dfrac{-3q^2-2pq+p^2}{3q^2+p^2}$.

To make these forms symmetrical, we may assume

$$p-q = k \text{ and } 2q = l, \text{ or } p = k+\tfrac{1}{2}l, \text{ and } q = \tfrac{1}{2}l.$$

Then we have

$$x = \frac{l^2-2kl-2k^2}{k^2+kl+l^2},\quad n' = \frac{2kl+l^2}{k^2+kl+l^2},\quad \text{and } n = \frac{-2kl-k^2}{k^2+kl+l^2} \text{ or } \frac{k^2-l^2}{k^2+kl+l^2},$$

and the second value of n must be taken to obtain a positive quantity, k and l being any positive numbers, k greater than l.

In the same way, from (2), we get

$$n' = \frac{2kl+l^2}{k^2+kl+l^2},\quad m' = \frac{k^2+2kl}{k^2+kl+l^2}.$$

We have thus for rational-sided triangles, or, clearing of fractions, for integral-sided triangles, the three sets of formulæ:

(1) $m = k^2+kl+l^2$, $n = k^2-l^2$; (2) $m = k^2+kl+l^2$, $n = 2kl+l^2$;

(3) $m = k^2+2kl$, $n = k^2+kl+l^2$;

and it is easy to see that in all three cases the area of the triangle is

$$(k^2+kl+l^2)\,(k+l)\,(k-l)\,(2k+l)\,(k+2l)\,kl.$$

E.g.—Putting $k = 2$, $l = 1$, we have

$$m = 7,\ n = 3, \text{ or } m = 7,\ n = 5, \text{ or } m = 8,\ n = 7,$$

giving the three equal right-angled triangles 40, 42, 58; 24, 70, 74; 15, 112, 113, respectively, the area in each case being 840.

[Lt.-Col. Allan Cunningham, R.E., remarks:—

This problem has been solved *with considerable completeness*, in Frenicle's *Traité des Triangles rectangles en nombres*, &c. (1729 ?). As this treatise covers 166 pages, only a *sketch* of the methods and a *few* of the results can be given here.

He gives a *general* direct process for generating *any number* of equal Pythagorean triangles from a *given* base-triangle. Let α_1, β_1, γ_1 be the sides of the base-triangle, such that

$$\gamma_1^2 = \alpha_1^2+\beta_1^2;$$

and write $\Sigma_1 = \frac{1}{2}\alpha_1\beta_1$, $\alpha_1^2 \sim \beta_1^2 = \delta_1$, $k_1 = 2\gamma_1\delta_1$;

and use the same symbols with subscripts 2, 3, ..., for the *derived* triangles.

Now take $\alpha_2 = \delta_1^2$, $\beta_2 = 4\alpha_1\beta_1\gamma_1^2$, $\gamma_2 = 4\alpha_1^2\beta_1^2+\gamma_1^4$.

Hence $\gamma_2^2 = \alpha_2^2+\beta_2^2$, $\Sigma_2 = \frac{1}{2}\alpha_2\beta_2 = 2\delta_1^2\alpha_1\beta_1\gamma_1^2 = k_1^2.\Sigma_1$.

Now form a new triangle,

$$a_1 = k_1\alpha_1,\quad b_1 = k_1\beta_1,\quad c_1 = k_1\gamma_1;$$

hence $c_1^2 = a_1^2+b_1^2$, $S_1 = \frac{1}{2}a_1b_1 = k_1^2.\Sigma = \Sigma_2$;

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hereby (a_1, b_1, c_1), (a_2, β_2, γ_2) are a *pair* of *equal* Pythagorean triangles $(S_1 = \Sigma_2)$.

Ex.—Take $a_1 = 3$, $\beta_1 = 4$, $\gamma_1 = 5$; $\Sigma_1 = 6$, as the base-triangle.

Then $\delta_1 = 7$, $k_1 = 70$.

Hence $a_1 = 210$, $b_1 = 280$, $c_1 = 350$; $a_2 = 49$, $\beta_2 = 1200$, $\gamma_1 = 1201$, are two *equal* Pythagorean triangles $(S_1 = 29400 = \Sigma_2)$.

This process can be *repeated*. Write

$$a_2^2 \sim \beta_2^2 = \delta_2, \quad k_2 = 2\gamma_2\delta_2.$$

Now take $a_3 = \delta_2^2$, $\beta_3 = 4a_2\beta_2\gamma_1^2$, $\gamma_3 = 4a_2^2\beta_2^2 + \gamma_2^4$;

Hence $\gamma_3^2 = a_3^2 + \beta_3^2$, $\Sigma_3 = \frac{1}{2}a_3\beta_3 = 2\delta_2^2 a_2\beta_2\gamma_2^2 = k_2^2.\Sigma_2 = k_2^2.k_1^2.\Sigma_1$.

Now form *two* new triangles,

$$a_1 = k_2k_1a_1, \quad b_1 = k_2k_1\beta_1, \quad c_1 = k_2k_1\gamma_1;$$

hence $c_1^2 = a_1^2 + b_1^2$;

and $a_2 = k_2a_2, \quad b_2 = k_2\beta_2, \quad c_2 = k_2\gamma_2;$

hence $c_2^2 = a_2^2 + b_2^2$.

Also $S_1 = \frac{1}{2}a_1b_1 = k_2^2k_1^2.\Sigma_1$, $S_2 = \frac{1}{2}a_2b_2 = k_2^2.\Sigma_2$;

therefore $S_1 = S_2 = \Sigma_2$;

Hence (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, β_3, γ_3) are a *triad* of *equal* Pythagorean triangles.

Ex.—Take $a_1 = 3$, $\beta_1 = 4$, $\gamma_1 = 5$; $\Sigma_1 = 6$ as base-triangle. Then

$$\delta_1 = 74, \quad k_1 = 70.$$

Hence $a_2 = 49$, $\beta_2 = 1200$, $\gamma_2 = 1201$, $\Sigma_2 = 29400$, as before.

Therefore $\delta_2 = 1437599$, $k_2 = 2.1201.1437599$.

Hence $a_3 = (1437599)^2$, $\beta_3 = 4.49.1200.1201^2$,

$$\gamma_3 = (2.49.1200)^2 + 1201^4.$$

From these data the three equal triangles (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, β_3, γ_3) can be readily formed.

The above process can evidently be *repeated ad infinitum*. It has the merit of being *direct*, but has the serious drawback of leading to enormous numbers (as seen above). The author, however, gives another *tentative* process, which amounts to *tabulating* the areas $\Sigma = \frac{1}{2}ab$ of a great number of Pythagorean triangles, and *selecting by inspection* those whose areas are in the ratio of $k_1^2 : 1$, $k_2^2k_1^2 : k_2^2 : 1$, &c. This would seem to be a very laborious process; but the author has carried it out with great success. He gives numerous examples—viz., 10 pairs, 9 triads, 3 quartets, 4 quintets, and 1 sextet. A *few* of these are given below—

Pairs. 20, 21, 29 / 12, 35, 37 } $S = 210$; 48, 55, 73 / 22, 120, 122 } $S = 1320$;

Triads. 40, 42, 58 / 24, 70, 74 / 15, 112, 113 } $S = 840$; 56, 390, 394 / 105, 208, 233 / 120, 182, 218 } $S = 10920$;

Quartets. (111, 6160, 6161), (231, 2960, 2969) / (280, 2442, 2458), (518, 1320, 1418) } $S = 341880$.

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Quintets. (2805, 52416, 52491), (3168, 46410, 46518)
(6006, 24480, 25206), (5236, 28080, 28564) } S = 73513440.
(8580, 17136, 19164)

The *sextet* is not worth printing, as the *smallest* side contains 15 figures.]

5019. (J. J. Walker, M.A.)—If A be the refracting angle of a prism, and θ, ϕ the differences between the angles which the part of the ray within the prism makes with the normals to the faces, and between the deviations at incidence and emergence respectively, while D is the total deviation, prove (1) that $\sin\frac{1}{2}(A+D) = \frac{m\cos\frac{1}{2}\theta\sin\frac{1}{2}A}{\cos\frac{1}{2}(\theta+\phi)}$, and thence (2) deduce the position of minimum deviation.

Solution by Professor Sanjána, M.A.

In the annexed figure

$\sin\alpha = \mu\sin\alpha'$, $\sin\beta = \mu\sin\beta'$;

also $\theta = \alpha' - \beta'$,

$\phi = \alpha - \alpha' - (\beta - \beta') = \alpha - \beta - (\alpha' - \beta')$
$= \alpha - \beta - \theta.$

As $\sin\alpha + \sin\beta = \mu(\sin\alpha' + \sin\beta')$,

therefore $2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
$= 2\mu\sin\frac{1}{2}(\alpha'+\beta')\cos\frac{1}{2}(\alpha'-\beta')$;

therefore $2\sin\frac{1}{2}(D+A)\cos\frac{1}{2}(\theta+\phi) = 2\mu\sin\frac{1}{2}A\cos\frac{1}{2}\theta$;

so that $\sin\frac{1}{2}(D+A) = \frac{\mu\cos\frac{1}{2}\theta\sin\frac{1}{2}A}{\cos\frac{1}{2}(\theta+\phi)}$.

If now $\alpha > \beta$, $\alpha' > \beta'$, and θ and ϕ are positive; but $\cos\frac{1}{2}\theta > \cos\frac{1}{2}(\theta+\phi)$, and hence $\sin\frac{1}{2}(D+A)$ is greater than $\mu\sin\frac{1}{2}A$. Thus D is not a minimum, unless ϕ vanish; which is the case when $\alpha = \beta$, or when the ray passes symmetrically, and then $\sin\frac{1}{2}(D+A) = \mu\sin\frac{1}{2}A$. If $\alpha < \beta$, reversing the path of the ray, the same conclusion will be obtained.

14115. (H. Fortey, M.A.)—A cheese is a cylinder of 2 feet in diameter, and is of variable quality, the part at the axis being worth 1s. 2d. per lb., and the quality deteriorating as the distance from the axis increases, so that at the circumference it is worth only 10d. per lb. An epicure buys a concentric cylindrical piece cut out of the centre and pays 4s. 11d. for it. What is the diameter of this piece? Given that the whole cheese is worth £1. 18s. 3d.

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Solution by Rev. J. H. BLUNN, Chaplain R.N., retired; H. W. CURJEL, M.A.; *and the* PROPOSER.

The decrease in value of the cheese is at the rate of $\frac{1}{3}$d. per inch of distance from centre. Therefore value per lb. at a distance x

$$= (14 - \tfrac{1}{3}x) \text{ d.}$$

Hence, evidently, if a be the radius of the cylinder cut out,

$$\int_0^a 2\pi x\,dx\,(14 - \tfrac{1}{3}x) \Big/ \int_0^{12} 2\pi x\,dx\,(14 - \tfrac{1}{3}x) = \tfrac{5}{4}\tfrac{9}{5}\tfrac{9}{5}.$$

Therefore $$(7a^2 - \tfrac{1}{9}a^3) / (7 \times 12^2 - \tfrac{1}{9} \times 12^3) = \tfrac{5}{4}\tfrac{9}{5}\tfrac{9}{5};$$

$$7a^2 - \tfrac{1}{9}a^3 = \tfrac{5}{4}\tfrac{9}{5}\tfrac{9}{5}(1008 - 192) = \tfrac{5}{4}\tfrac{9}{5}\tfrac{9}{5} \times 816 = \tfrac{1}{9}(59 \times 16).$$

Therefore $a^3 - 63a^2 + 59 \times 16 = 0$, $(a-4)(a^2 - 59a - 236) = 0$.

$a = 4$ is the only possible solution. Therefore

diameter of the piece = 8 inches.

14110. (Professor SANJÁNA, M.A.)—In a triangle ABC, K is the symmedian point and O the circumcentre; on KA, KB, KC points A′, B′, C′ are taken such that KA′ : KA = KB′ : KB = KC′ : KC = e (which may be positive or negative and less than unity or greater); through A′, B′, C′ are drawn Y′A′Z, Z′B′X, X′C′Y anti-parallels to BC, CA, AB meeting these sides in X and X′, Y and Y′, Z and Z, respectively. Prove (1) that $Y'Z = Z'X = X'Y = 2(1-e)\,abc/(a^2 + b^2 + c^2)$; (2) that YZ′ is parallel to BC and $= a\left\{(2e-1)\,a^2 + b^2 + c^2\right\}/(a^2 + b^2 + c^2)$, with similar results for ZX′ and XY′; (3) that

$$XX' = a\left\{a^2 + (b^2 + c^2)\,(2e-1)\right\}/(a^2 + b^2 + c^2),$$

with similar values for YY′ and ZZ′; (4) that the triangles XYZ, X′Y′Z are equal to each other and similar to ABC; and that XYZ and ABC have the same positive BROCARD point, and X′Y′Z′ and ABC the same negative one; (5) that

$AZ : BX : CY = b/a : c/b : a/c$ and $AY' : CX' : BZ' = c/a : b/c : a/b$;

(6) that the points X, X′, Y, Y′, Z, Z′ are situated on a circle whose centre lies on OK and coincides with the circumcentre of A′B′C′; (7) that the radius of this circle is $R\left\{e^2 + (1-e)^2 \tan^2\omega\right\}^{\frac{1}{2}}$; (8) that for values 1, $\frac{1}{2}$, 0 of e the circle XX′YY′ZZ′ becomes respectively the circumcircle, the triplicate-ratio circle, the cosine circle; (9) that when

$$e = -\cos A \cos B \cos C,$$

the circle becomes the TAYLOR circle, and its radius

$$= R(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C)^{\frac{1}{2}};$$

(10) that when $e = \sin^2\omega$, the centre of the circle bisects the line joining the BROCARD points, and its radius $= R \sin\omega$, which is the least value that the radius can ever have; and (11) that in a right-angled

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triangle the TAYLOR and cosine circles coincide. [All properties of TUCKER circles can thus be obtained by elementary trigonometry; the result in (11) is inaccurately given in MILNE'S *Companion* at p. 146.]

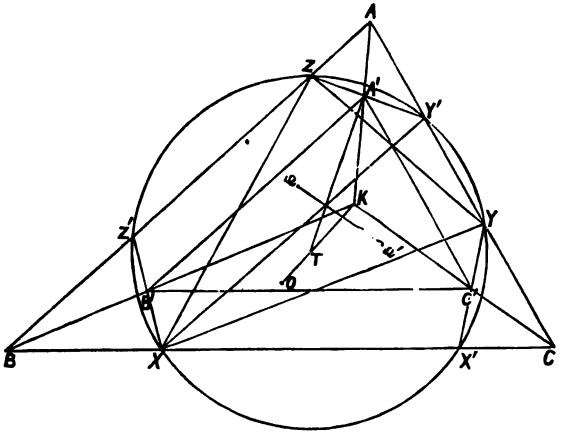
Solution by G. N. BATES.

We have $BZ'/a = BX/c = Z'X/b = \rho$

and $4BB'^2 = 2BZ'^2 + 2BX'^2 - Z'X^2$;

therefore $\rho^2 = \{4(1-e)^2 a^2c^2\} / (a^2+b^2+c^2)^2$;

therefore $Z'X = 2(1-e)abc / (a^2+b^2+c^2) = X'Y = Y'Z$ ..... (1).



$CY : CA :: a\rho : c :: BZ' : AB$;

therefore YZ' is parallel to BC, and

$YZ'/a = (c - a\rho)/c = \{a^2(2e-1) + b^2 + c^2\} / (a^2+b^2+c^2)$ ..... (2);

$XX' = a - c\rho - (b^2/c)\rho = a\{(2e-1)(b^2+c^2) + a^2\} / (a^2+b^2+c^2)$... (3).

Now ZZ', $Y'Y$ are concyclic, and

$\angle Z'XY' = \angle C = \angle Z'YY'$, $\angle ZX'Y = \angle B = \angle ZZ'Y$;

therefore XX', YY', ZZ' are concyclic. If R_e be the radius of this circle,

$$\frac{ZY}{\sin B} = \frac{YX}{\sin A} = \frac{XZ}{\sin C} = \frac{Z'Y'}{\sin C} = \frac{Y'X'}{\sin B} = \frac{X'Z'}{\sin A} = 2R_e;$$

therefore triangles XYZ, $X'Y'Z'$ are similar to one another and to ABC, and, being inscribed in the same circle, are equal.

Also, in triangles ΩZA, ΩBX,

$\angle \Omega AZ = \omega = \angle \Omega BX$ and $\Omega A : AZ = \Omega B : BX$;

therefore $\angle Z\Omega A = \angle B\Omega X$;

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therefore $\angle Z\Omega X = A\Omega B$ and $\angle \Omega XY = \omega$;

therefore Ω is positive BROCARD point of triangle XYZ; similarly, Ω' is negative BROCARD point of triangle X'Y'Z' ..... (4).

$$AZ : BX : CY = (b^2/a)\rho : c\rho : (ab/c)\rho = b/a : c/b : a/c;$$

$$AY' : CX' : BZ' = bc\rho/a : b^2\rho/c : a\rho = c/a : b/c : a/b \quad \ldots\ldots (5).$$

Z'X, X'Y, Y'Z are equal chords of a circle; and A', B', C', their mid-points, therefore lie on a concentric circle. If A'T be drawn parallel to AO meeting OK in T, T is obviously the centre of A'B'C', and

$$KT : KO = e : 1 \quad \ldots\ldots (6).$$

$$ZX^2 = BZ^2 + BX^2 - 2\,BZ \cdot BX \cos B;$$

therefore $a^2ZX^2 = \rho^2 \Sigma a^2b^2 + abc \Sigma a^2 \cdot \rho + a^2c^2;$

therefore $ZX^2 = c^2\left\{e^2 + (1-e)^2 \tan^2\omega\right\};$

therefore $R_e = R\left\{e^2 + (1-e)^2 \tan^2\omega\right\} \quad \ldots\ldots (7).$

When $e = 1$, T is at O,

$$\rho = 0, \quad XX' = a, \quad R_e = R;$$

therefore circle is circumcircle. When $e = \frac{1}{2}$, T is mid-point of OK, and

$$XX' = a^3/(a^2 + b^2 + c^2);$$

therefore circle is triplicate-ratio circle. When $e = 0$, T is at K, and

$$XX' : YY' : ZZ' = \cos A : \cos B : \cos C;$$

and circle is cosine circle ..... (8).

When $$e = -\cos A \cos B \cos C,$$

$$CX = a - c\rho = a\cos^2 C,$$

and X is projection on BC of that vertex of pedal triangle which lies on AC, &c.; therefore circle is TAYLOR circle, and

$$Re = R(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C)^{\frac{1}{2}} \quad \ldots\ldots (9).$$

If $e = \sin^2\omega$, $KT = KO \sin^2\omega$, $K\Omega = OK \sin\omega$;

therefore $KT/K\Omega = \sin\omega$,

and T is at mid-point of $\Omega\Omega'$. Now

$$(1-e)^2 \tan^2\omega + e^2 = e^2 \sec^2\omega - 2e \tan^2\omega + \tan^2\omega,$$

which is a minimum when $e = \sin^2\omega$, and in this case

$$R_e = R \sin\omega \quad \ldots\ldots (10).$$

In a right-angled triangle,

$$\cos A \cos B \cos C = 0,$$

and TAYLOR and cosine circles coincide ..... (11).

13926. (Professor A. DROZ-FARNY.)—Sur une parabole on donne un point A, la tangente en ce point, et la droite qui joint A au foyer F. Une tangente variable coupe la tangente en A au point B, et AF en C. On prend sur cette droite le symétrique D de B par rapport à C. Lieu géométrique de D? On demande une solution analytique et une solution géométrique.

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Solution by R. F. Davis, M.A.

The exterior angle ABC between two tangents is equal to the angle AFB, subtended at the focus by one of them (Milne and Davis, p. 30). Thus

$$CB^2 = CA \cdot CF.$$

But $CB = CD$;

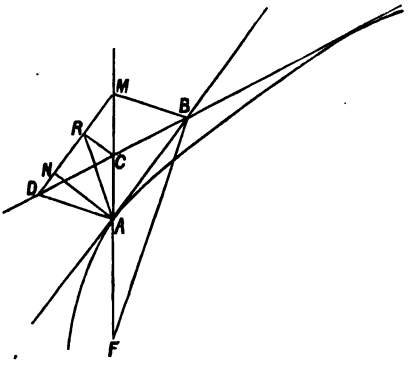
hence $CD^2 = CA \cdot CF$.

Completing the parallelogram BADM, and drawing perpendiculars AN, CR upon DM,

$$CR^2 + DR^2 = CA^2 + CA.AF,$$

$$DR^2 = RM^2 + CA.AF$$

$$= \lambda AR^2 + \mu AR,$$



for the right-angled triangle AMN is of fixed species, and AM, AN have fixed directions, and therefore so has AR.

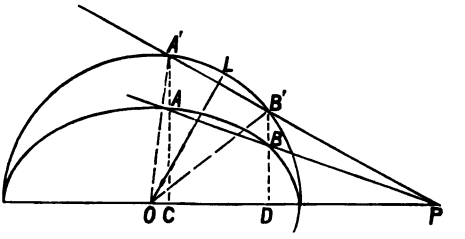
The locus of D is thus a hyperbola touching the given parabola at A.

14096. (G. W. Preston, B.A.)—If a chord of an ellipse make a constant angle with the major axis, show geometrically that the sum of the excentric angles of its extremities is a constant.

Solution by Professor A. Droz-Farny; J. O. Watts; *and many others.*

Soient A et B deux points d'une ellipse tels que la droite AB forme avec le grand axe de la courbe un angle constant.

Les ordonnées AC et BD de ces points rencontrent en A′ et B′ le cercle sur le grand axe comme diamètre et les angles excentriques sont respectivement



$$A'OC = \alpha; \quad B'OD = \beta.$$

On sait que les cordes A′B′ et AB se coupent en P sur le grand axe.

Le triangle ACP étant de forme donnée, on a

$$AC/CP = \text{constant}.$$

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Mais $A'C/AC = a/b$; donc $A'C/CP$ = constant.

Le triangle A'CP est donc aussi de forme donnée, donc le côté A'B' est de direction fixe, et par conséquent aussi la droite OL perpendiculaire à A'B'; donc $\angle LOP = \frac{1}{2}(\alpha+\beta)$ = constant.

[Mr. CUBJEL observes:—The point of contact of the tangent parallel to the chord is fixed, and its excentric angle = ½ the sum of the excentric angles of the ends of the chord, which is therefore constant.]

14103. (Prof. CROFTON, F.R.S.)—If $u = x^2+ax+b$, a solution of the equation $u(d^2y/dx^2) = n(n+1)y$ is $y = D^{n-1}u^n$; and another solution is $y = D^{-n-2}u^{-n-1}$. Also, if $u^2(d^4y/dx^4) = n(n+1)(n+2)(n+3)y$, one solution is $y = D^{n-1}u^{n+1}$.

Solution by Rev. J. CULLEN, B.A.

If $u = x^2+ax+b = (x-\alpha)(x-\beta)$, we have, by a well known theorem (WILLIAMSON's *Int. Cal.*, p. 325, 5th ed.),

$$u^m D^{m+r}u^r = \frac{(r+m)!}{(r-m)!} D^{r-m}u^r.$$

Putting $r = m+n-1$, we get

$$u^m D^{2m} D^{n-1}u^{n+m-1} = n(n+1)\ldots(n+2m-1)\,D^{n-1}u^{n+m-1},$$

so that clearly $y = D^{n-1}u^{n+m-1}$ is one solution of

$$u^m D^{2m} y = n(n+1)\ldots(n+2m-1)\,y \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also, since (1) remains unaltered when $-(n+2m-1)$ is written for n, we have another solution in the form $y = D^{-n-2m}u^{-n-m}$. Hence the results in the question follow at once by putting $m = 1$ and 2.

14117. (G. HALE PUCKLE, M.A.)—By the methods of Quest. 13535, No. 437, show that the equation to a conic, $\phi(xy) = 0$, is equivalent to

$$\rho^2\left\{\alpha(ax+hy+g)+\beta(hx+by+f)+\Delta^{\frac{1}{2}}\right\}^2 = (Cx-G-a\Delta^{\frac{1}{2}})^2+(Cy-F-\beta\Delta^{\frac{1}{2}})^2.$$

Solution by the PROPOSER.

From the equations for the foci we have

$$x' = (G+a\Delta^{\frac{1}{2}})/C, \quad y' = (F+\beta\Delta^{\frac{1}{2}})/C,$$

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and the equation to the conic is

$$\{ax+\beta y+(g\alpha+f\beta+\Delta^{\frac{1}{2}})/\phi^2\}^2=\rho^2(x-x')^2+\rho^2(y-y')^2;$$

whence, since $C=-\phi^2\rho^2$, the result is obtained.

Similarly (Quest. 13794, No. 448) the equation may be written

$$\phi^2\{\beta(ax+hy+g)-\alpha(hx+by+f)-i\Delta^{\frac{1}{2}}\}^2 = (Cx-G-\beta i\Delta^{\frac{1}{2}})^2+(Cy-F+\alpha i\Delta^{\frac{1}{2}})^2.$$

14167. (Professor NEUBERG.)—Soient A_1, B_1, C_1, ... les mineurs du déterminant $H=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Démontrer que $(B_1C_2A_3-B_2C_3A_1)=H(b_1c_2a_3-b_2c_3a_1)$.

Solutions (1) *by* Professor JAN DE VRIES *and* G. BIRTWISTLE, B.Sc.;
(2) *by the* PROPOSER.

(1) Denoting by α_1, β_1, ... the minors of the reciprocal determinant formed by A_1, B_1, ..., we have

$$a_1=a_1H \text{ or } B_2C_3-B_3C_2=a_1H,\quad \gamma_2=c_2H \text{ or } B_1A_3-B_3A_1=c_2H.$$

Multiplying these relations respectively by $-A_1=b_3c_2-b_2c_3$ and $C_2=b_1a_3-b_3a_1$, we get

$$B_1C_2A_3-B_2C_3A_1=H(b_1c_2a_3-b_2c_3a_1).$$

(2) (i.) On fait le produit de déterminant H par le déterminant

$$\begin{vmatrix} 0 & -c_1 & b_1 \\ c_2 & 0 & -a_2 \\ -b_3 & a_3 & 0 \end{vmatrix}=b_1c_2a_3-b_2c_3a_1, \ldots.$$

(ii.) Si l'on considère (a_1, b_1, c_1), ... comme les coordonnées des sommets d'un triangle A'B'C' homologique avec le triangle de référence ABC, la condition que les droites AA', BB', CC' concourent en un même point s'exprime par l'équation $b_1c_2a_3-b_2c_3a_1=0$. Mais, si l'on écrit que les droites B'C', C'A', A'B', représentées par les équations

$$A_1x+B_1y+C_1z=0, \ldots,$$

rencontrent respectivement BC, CA, AB en trois points d'une même droite, on trouve $B_1C_2A_3-B_2C_3A_1=0$. On sait, par la géométrie, que les deux propriétés des triangles ABC, A'B'C' sont l'une la conséquence de l'autre; donc la quantité $B_1C_2A_3-B_2C_3A_1$, qui est du sixième degré en a_1, b_1, ... est divisible par $b_1c_2a_3-b_2c_3a_1$. Le quotient est nécessairement du troisième degré. Mais, lorsque les points A', B', C' sont en ligne droite, la condition que les points de rencontre des droites (BC, B'C'), (CA, C'A'), (AB, A'B') soient en ligne droite est encore remplie; mais alors $H=0$. Donc H doit être le second facteur de $B_1C_2A_3-B_2C_3A_1$.

Nous avons donné cette démonstration dans *Mathesis*, 1898, p. 195;

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avant de connaître les recherches de T. MUIR, nous étions déjà en po sion de la propriété qui fait l'objet de la Quest. 14167.

[Dr. T. MUIR, in a paper read before the Royal Society of Edinburgh on December 3, 1894, and entitled "On a Theorem regarding the Difference between any two Terms of the Adjugate Determinant," has shown that, in the case of determinants of the third order, this difference is divisible by the original determinant, the quotient being the difference of the corresponding terms of the original determinant.]

13980. (R. W. D. CHRISTIE.) — Find a general value for a so that

$$a.10^n - 1 \equiv P\,(M),$$

where P is any odd prime.

Solution by the PROPOSER.

Let

$$u_n = m^{n-1} + n - 2\,.\,m^{n-3} + \frac{n-3\,.\,n-4}{2!}\,m^{n-5} + \frac{n-4\,.\,n-5\,.\,n-6}{3!}\,m^{n-7} + \ldots$$

Then

$$(10u_n + u_{n+1})\,10^n - 1$$

$$= (10^2 + 10m - 1)(u_n 10^{n-1} + u_{n-1}10^{n-2} + u_{n-2}10^{n-3} + \ldots + 1)$$

$$= 10^{n+1}u_n + 10^n u_{n-1} + 10^{n-1}u_{n-2} + 10^{n-2}u_{n-3} + 10^{n-3}u_{n-4} + 10^{n-4}u_{n-5} + \ldots$$
$$+ m\,.\,10^n u_n + m\,.\,10^{n-1}u_{n-1} + m\,.\,10^{n-2}u_{n-2} + m\,.\,10^{n-3}u_{n-3}$$
$$+ m\,.\,10^{n-4}u_{n-4} + \ldots$$
$$- 10^{n-1}u_n - 10^{n-2}u_{n-1} - 10^{n-3}u_{n-2} - 10^{n-4}u_{n-3} + \ldots - 1$$

$$= 10^{n+1}u_n + 10^n(u_{n-1} + m\,.\,u_n) + 10^{n-1}(u_{n-2} + m\,.\,u_{n-1} - u_n)$$
$$+ 10^{n-2}(u_{n-3} + m\,.\,u_{n-2} - u_{n-1}) + \ldots - 1.$$

Now

$$u_{n-r} + m\,.\,u_{n-r+1} - u_{n-r+2} = 0$$

by the nature of the series (which is an extension of the "continuant" series).

Thus $(10u_n + u_{n+1})\,10^n - 1 = 10^{n+1}u_n + 10^n(u_{n-1} + m\,.\,u_n) - 1,$

i.e.,

$$10^n u_{n+1} = 10^n(u_{n-1} + m\,.\,u_n)$$
$$u_{n+1} = u_{n-1} + m\,.\,u_n,$$

which is true by the nature of the series.

Thus the general value for a is $10u_n + u_{n+1}$.

Also P is a factor of $10m + 99$, which is either a prime or contains any odd prime except 5 as a factor.

Also (M) consists of the terms of the series arranged in reverse order.

Ex. 1.—Let $m = 1$, $n = 5$; then we have the "continuant" series 1, 1, 2, 3, 5, &c., and $(10 \times 5 + 8)\,10^5 - 1 = 109 \times 53211$ (*v.* Quest. 14036, 14040, &c.).

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14128. (G. W. PRESTON, B.A.)—Show that

(i.) $\dfrac{\Sigma(y-z)^8}{\Sigma(y-z)^2} - 4\Pi(y-z)^2 = \{\Sigma x^2 - \Sigma yz\}^3$;

(ii.) $\underset{abcd}{\Sigma} \dfrac{b+c+d}{(b-a)(c-a)(d-a)(x-a)} = \dfrac{a+b+c+d-x}{(x-a)(x-b)(x-c)(x-d)}$.

[See CHRYSTAL'S *Algebra*.]

Solution by LIONEL E. REAY, B.A.; *the* PROPOSER; Professor SANJÁNA; *and others.*

(i.) Put $y-z=a$, $z-x=b$, $x-y=c$, &c.; then $a+b+c=0$.

Also $\Sigma a^2 = \Sigma(y-z)^2 = 2\{\Sigma x^2 - \Sigma yz\}$ and $\Sigma a^2 = -2\Sigma bc$;

therefore $4(\Sigma bc)^2 = (\Sigma a^2) = \Sigma a^4 + 2\Sigma b^2c^2$.

But $(\Sigma bc)^2 = \Sigma b^2c^2 + 2abc\Sigma a = \Sigma b^2c^2$;

$\therefore\ \Sigma a^4 = 2\Sigma b^2c^2$; $\therefore\ (\Sigma a^2)^2 = 2\Sigma a^4$; $\therefore\ (\Sigma a^4)^2 = 4(\Sigma b^2c^2)^2$;

$\therefore\ \Sigma a^8 = 2\Sigma b^4c^4 + 8(\Sigma a^2)a^2b^2c^2$; $\therefore\ (\Sigma a^4)^2 = 2\Sigma a^8 - 8a^2b^2c^2\Sigma a^2$;

$\therefore\ (\Sigma a^2)^4 = 4(\Sigma a^4)^2 = 8\Sigma a^8 - 32a^2b^2c^2\Sigma a^2$;

$\therefore\ (\Sigma a^2)^3 = 8(\Sigma a^8/\Sigma a^2) - 32a^2b^2c^2$;

$\therefore\ \{\Sigma x^2 - \Sigma yz\}^3 = \{\Sigma(y-z)^8/\Sigma(y-z)^2\} - 4\Pi(y-z)^2$.

(ii.) Bringing to a common denominator, the numerator becomes

$$\Sigma(b+c+d)(b-c)(b-d)(c-d)(x-d)(x-b)(x-c),$$

and the denominator

$$(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(x-a)(x-b)(x-c)(x-d).$$

Putting $b=c$ in the numerator, it vanishes; therefore $b-c$ is a factor.

Similarly for $c-a$, &c.; therefore $(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$ is a factor.

Putting $x=\Sigma a$, the numerator becomes $\Pi(b+c+d)\Sigma(c-a)(a-d)(c-d)$.

But $\Sigma(c-a)(a-d)(c-d)=0$; therefore $x-\Sigma a$ is a factor.

The remaining factor is numerical. To find it, put $x=0$; then

$$-\Sigma(b+c+d)(b-c)(b-d)(c-d)bcd \equiv -P\Sigma a\Pi(b-c),$$

where P is numerical; equating coefficients of b^4c^2d, $P=-1$. Therefore

$$\text{expression} = (\Sigma a - x)/\{(x-a)(x-b)(x-c)(x-d)\}.$$

[Professor SANJÁNA observes that the result is inaccurately given in CHRYSTAL'S *Algebra*.]

14099. (H. W. CURJEL, M.A.)—In a given quadrilateral inscribe a quadrilateral similar to a given quadrilateral.

Solution by C. E. HILLYER, M.A.

Let ABCD be the quadrilateral in which a quadrilateral is to be inscribed similar to *abcd*.

Let DA, CB meet in E, and AB, DC in F.

On AB describe a segment of a circle containing an angle equal to $b-E$, and on AE a segment containing an angle equal to $EBA-bac$,

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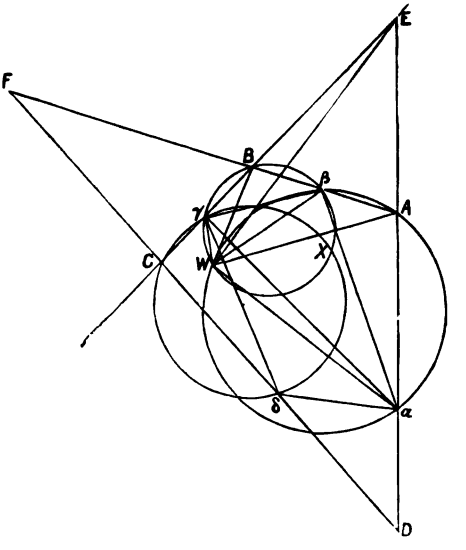
and let the two circles meet in W. Similarly, let X be the intersection of circles on FB and on CB, such that $FXB = FCB - d$ and $CXB = c - F$.

Let the circle XBW meet AB in β and BC in γ, the circle WβA meet AD in α, and the circle XγC meet CD in δ; then $\alpha\beta\gamma\delta$ shall be the quadrilateral required.



For $\angle \alpha\beta\gamma = E\gamma\beta + E\alpha\beta + E = BW\beta + AW\beta + E = BWA + E = b$.

Also, since $\angle \gamma W\alpha = \gamma W\beta + \alpha W\beta = EBA + EAB$,

E, γ, W, α are concyclic, and

$$\angle \beta\alpha\gamma = \beta\alpha W - \gamma\alpha W = BAW - BEW = BAD - WAD - E + WEA = EBA - EWA = bac;$$

therefore $\alpha\beta\gamma$ is similar to abc; and in the same way it can be shown that $\gamma\delta\alpha$ is similar to cda.

14189. (V. Daniel.)—XY is the line $x\cos\alpha + y\sin\alpha = p$, referred to rectangular axes through O. If a point move along the zigzag line $OPP_1P_2P_3P_4\ldots$ (which is such that OP is perpendicular to XY, PP_1 to YO, and, generally, P_nP_{n+1} perpendicular to $P_{n-2}P_{n-1}$), it will ultimately arrive at a point S. Similarly, if it move along $OPp_1p_2p_3\ldots$, it will

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ultimately arrive at a point S′. Prove (1) S and S′ are the foci of an ellipse inscribed in the triangle YOX, and its major axis is

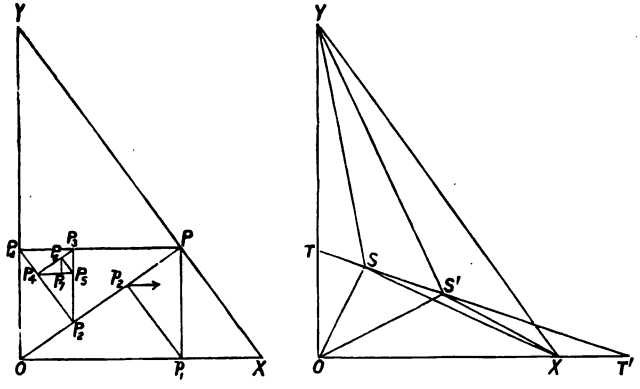
$$p/\sqrt{(1+\cos^2\alpha\sin^2\alpha)}.$$

(2) OX, OY subtend right angles at S and S′ respectively, and hence that three pairs of triangles are similar, namely, OSX, OS′Y; OSY, XS′Y; OS′X, XSY. (3) If SS′ produced intersect the axes in T and T′, $OS^3.OT' = OS'^3.OT$. (4) When YOX is isosceles, the ellipse has eccentricity $= 1/\sqrt{5}$.

Solution by the PROPOSER, *and* R. TUCKER, M.A.

Continue the line $OPP_1P_2\ldots$ until the triangle $P_5P_6P_7$ is formed, similar and similarly situated to the triangle OXY. The coordinates of P_7 are $x = P_1P_3 - P_5P_7$, $y = OP_1 - P_3P_5$. Since all the triangles in the figure are similar, these are evidently

$$x = p\cos\alpha\sin^2\alpha - p\cos^3\alpha\sin^4\alpha = p\cos\alpha\sin^2\alpha(1-\cos^2\alpha\sin^2\alpha),$$
$$y = p\sin\alpha - p\sin^3\alpha\cos^2\alpha = p\sin\alpha(1-\cos^2\alpha\sin^2\alpha).$$



Regarding $P_5P_6P_7$ as the original triangle, the coordinates (referred to P_7 as origin) of the right angle of the next similar and similarly-situated triangle are the same as x and y with p_1 written for p; so that the coordinates of the limiting point at which the triangle vanishes are given by

$$x = \cos\alpha\sin^2\alpha(1-\cos^2\alpha\sin^2\alpha)\{p+p_1+p_2+\ldots+p_n\}_{n=\infty},$$
$$y = \sin\alpha(1-\cos^2\alpha\sin^2\alpha)\{p+p_1+p_2+\ldots+p_n\}_{n=\infty}.$$

But $$p_n/p_{n-1} = p_{n-1}/p_{n-2} = \ldots = p_1/p = \cos^4\alpha\sin^4\alpha.$$

Hence $$\{p+p_1+p_2+\ldots+p_n\}_{n=\infty} = p/(1-\cos^4\alpha\sin^4\alpha).$$

Therefore S is the point;

$$x = \frac{p\cos\alpha\sin^2\alpha}{1+\cos^2\alpha\sin^2\alpha}; \qquad y = \frac{p\sin\alpha}{1+\cos^2\alpha\sin^2\alpha}.$$

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Similarly, S′ is the point ;

$$x' = \frac{p\cos\alpha}{1+\cos^2\alpha\sin^2\alpha}; \quad y' = \frac{p\sin\alpha\cos^2\alpha}{1+\cos^2\alpha\sin^2\alpha}.$$

(1) If z and z' are the perpendiculars from S and S′ on XY,

$$-z = \left(\frac{p\cos\alpha\sin^2\alpha}{1+\cos^2\alpha\sin^2\alpha}\right)\cos\alpha + \left(\frac{p\sin\alpha}{1+\cos^2\alpha\sin^2\alpha}\right)\sin\alpha - p = \frac{-p\cos^2\alpha}{1+\cos^2\alpha\sin^2\alpha};$$

$$-z' = \left(\frac{p\cos\alpha}{1+\cos^2\alpha\sin^2\alpha}\right)\cos\alpha + \left(\frac{p\sin\alpha\cos^2\alpha}{1+\cos^2\alpha\sin^2\alpha}\right)\sin\alpha - p = \frac{-p\sin^2\alpha}{1+\cos^2\alpha\sin^2\alpha}.$$

Hence $$xx' = yy' = zz' = \frac{p^2\cos^2\alpha\sin^2\alpha}{(1+\cos^2\alpha\sin^2\alpha)^2},$$

which is a property of the focal perpendiculars on the tangents to an ellipse. If a and b are its semi-axes, and e is its eccentricity,

$$b^2 = \frac{p^2\cos^2\alpha\sin^2\alpha}{(1+\cos^2\alpha\sin^2\alpha)^2}$$

and $$4a^2e^2 = SS'^2 = (x-x')^2 + (y-y')^2$$

$$= \frac{p^2(\cos^6\alpha+\sin^6\alpha)}{(1+\cos^2\alpha\sin^2\alpha)^2} = \frac{p^2(1-3\cos^2\alpha\sin^2\alpha)}{(1+\cos^2\alpha\sin^2\alpha)^2};$$

therefore $$4a^2 = 4b^2 + 4a^2e^2 = p^2/(1+\cos^2\alpha\sin^2\alpha).$$

(2) The m of the line $OS = y/x = 1/(\cos\alpha\sin\alpha)$.

X is the point $(p\sec\alpha, 0)$. Therefore the m of the line XS

$$= y/(x-p\sec\alpha) = -\cos\alpha\sin\alpha.$$

$\angle OSX$ is therefore $\frac{1}{2}\pi$, and, by symmetry, $\angle OS'Y = \frac{1}{2}\pi$.

It follows at once that the three triangles are similar, since the focal distances of a point make equal angles with the tangents drawn from that point to the ellipse.

(3) $$\frac{OS}{OS'} = \left\{\frac{x^2+y^2}{x'^2+y'^2}\right\}^{\frac{1}{2}} = \tan\alpha, \quad \frac{OT}{OT'} = -\frac{y-y'}{x-x'} = \tan^3\alpha;$$

whence $$OS^3.OT' = OS'^3.OT.$$

(4) From (1), $$e^2 = \frac{1-3\cos^2\alpha\sin^2\alpha}{1+\cos^2\alpha\sin^2\alpha}.$$

Hence $$e = 1/\sqrt{5}, \quad \text{when} \quad \alpha = \tfrac{1}{4}\pi.$$

14204. (J. Macleod, M.A.)—B, E are the remote points in which the line of centres of two unequal circles which intersect meets the circumferences; A one of the points of intersection of the circles; BF = BA; ED = EA; $\angle FAC = \frac{1}{2}\angle BAE$. The radius AO produced meets the circumference in G; AD produced meets the same circumference in H, and AC meets in K. Prove AKGH a parallelogram.

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Solution by Professor T. SAVAGE.

Since $\angle H$ is right, it is sufficient to show $\angle DAC$ right. Produce BA to W.

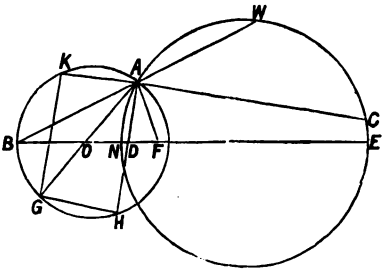
$\angle BAF + \angle EAD = \angle DFA + \angle FDA$, (Euc. I. 5)

$\therefore$ $\angle BAE + 2\angle DAF$ = two right angles;

$\therefore$ $\angle DAF = \frac{1}{2}\angle EAW$.

But $\angle FAC = \frac{1}{2}\angle BAE$,

$\therefore$ $\angle DAC$ = right angle.



14133. (Lt.-Col. ALLAN CUNNINGHAM, R.E.) — Find a sum of successive odd cubes equal to a square.

Solution by the PROPOSER.

Let S_n = sum of successive cubes $[1^3$ to $n^3]$; $S_n = \left(n\frac{n+1}{2}\right)$;

let s_{2r} = sum of successive *even* cubes $[1^3$ to $(2r)^3]$;

let s_{2r-1} = sum of successive *odd* cubes $[1^3$ to $(2r-1)^3]$.

Then $s_{2r} = 2^3(1^3+2^3+\ldots+r^3) = 2^3 S_r$,

and $s_{2r-1} = S_{2r} - s_{2r} = \left(2r\frac{2r+1}{2}\right)^2 - 2^3\left(r\frac{r+1}{2}\right)^2$

$= r^2(2r^2-1) = r^2.z^2$ (if $s_{2r-1} = \square$).

Hence every solution of $2r^2-z^2=1$ gives a solution of the problem; the solutions r, z are known to be as shown below; the *last* term of the series is $(2r-1)$.

r = 1, 5, 29, 169, 985, 5741, 33461, &c.,

z = 1, 7, 41, 239, 1393, 8119, 47321, &c.,

$2r-1$ = 1, 9, 57, 337, 1969, 11481, 66931, &c.

Thus the *lowest* solution is $1^3+3^3+5^3+7^3+9^3 = (517)^2$.

As to any *other* solutions not beginning (as above) with 1:

Let s = sum of successive odd cubes $(2\rho+1)^3+(2\rho+3)^3+\ldots+(2r-1)^3$,

so that $s = s_{2r-1} - s_{2\rho-1} = r^2(2r^2-1) - \rho^2(2\rho^2-1)$

$= (r^2-\rho^2)(2r^2+2\rho^2)+1$.

If $s = \square$, each of its factors, being prime to one another, must be squares. Therefore $r-\rho = t^2$, $r+\rho = u^2$, where

$$2r^2+2\rho^2+1 = t^4+u^4+1 = \square,$$

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an *impossible* result; see EULER's *Comment. Arithm.*, Petrop., 1849, Vol. II., p. 281. Hence the system of solutions given above is the only one.

[Mr. WHITWORTH solves the Question as follows:—Let N/D, where D is an odd number, be one of the convergents obtained by expressing $\sqrt{2}$ as a continued fraction. Then

$$1^3 + 3^3 + 5^3 + \ldots + (2D-1)^3 = N^2D^2.$$

Thus we have a series of consecutive odd cubes whose sum is a square.

The simplest case is when N = 5, D = 7, thus:

$$1^3 + 3^3 + 5^3 + 7^3 + 9^3 = 35^2.]$$

14095. (R. KNOWLES, B.A.)—

$$ax^2 + bxy + cy^2 + ey = 0, \quad a'x^2 + b'xy + c'y^2 + e'y = 0$$

are the equations to two parabolas, touching at P, referred to a tangent and normal at P as axes; if F, F′ are the foci, prove that, if $b : c = b' : c'$, F, F′, P are collinear.

Solution by R. TUCKER, M.A.; Prof. A. DROZ-FARNY; *and many others.*

Let the curve cut the normal in A. Therefore

$$AP = -e/c \quad \ldots\ldots \text{(i.)}.$$

Since the curve is a parabola,

$$b^2 = 4ac;$$

and tangent at A is given by

$$bx + cy + e = 0;$$

therefore $PR = -e/b$ ..... (ii.).

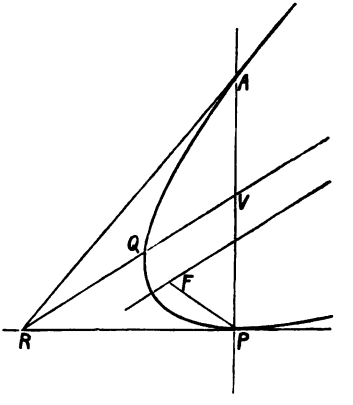
Therefore

$$\tan PRV = PV/PR = b/2c = \tan RPF.$$

Hence, if $b : c = b' : c'$,

then $\angle RPF = RPF'$;

Therefore &c.



14160. (G. BIRTWISTLE, B.Sc.) — The tangents to any umbilical geodesic on a quadric all intersect the focal hyperbola.

Solution by Rev. J. CULLEN.

For a geodesic we have, in ordinary notation,

$$a'^2 \cos^2 i + a'^2 \sin^2 i = k^2,$$

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where $$k^2 = a^2(1 - b^2c^2/p^2D^2).$$

Now pD is constant and equals ac at an umbilic; therefore

$$a'^2\cos^2 i + a'^2\sin^2 i = a^2 - b^2.$$

But, by a known theorem, tangents to a geodesic touch the confocal whose primary axis is k, which reduces to the focal hyperbola

$$x^2/(a^2 - b^2) + z^2/(c^2 - b^2) = 1,$$

when $k^2 = a^2 - b^2$; therefore, &c.

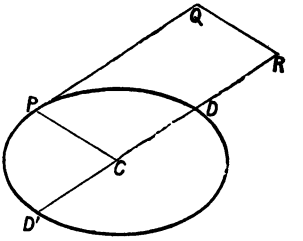
14093. (Rev. T. Mitcheson, B.A.)—PQ, a tangent to an ellipse at P, is equal to a diameter parallel to itself: find the locus of Q.

Solution (1) *by* J. O. Watts; R. Tucker, M.A.; *and others*; (2) *by* H. A. Webb; H. W. Curjel, M.A.; *and others.*

(1) Let P be $(a\cos\alpha,\ b\sin\alpha)$. Let D be $(-a\sin\alpha,\ b\cos\alpha)$. Let

$$CR = 2CD.$$

Then R is $(-2a\sin\alpha,\ 2b\cos\alpha)$. Therefore, if RQ is parallel to PC, meeting tangent in Q, Q is the moving point.



RQ is $$\frac{x + 2a\sin\alpha}{y - 2b\cos\alpha} = \frac{a\cos\alpha}{b\sin\alpha},$$

or $$(x/a)\sin\alpha - (y/b)\cos\alpha = -2$$

PQ is $$(x/a)\cos\alpha + (y/b)\sin\alpha = 1$$

Therefore locus is (squaring and adding) $x^2/a^2 + y^2/b^2 = 5$.

(2) PQ is equal and parallel to DD'. We have to find the locus of Q.

Project the ellipse orthogonally into a circle, and we have at once that the locus of the projection of Q is a concentric circle whose radius is $\sqrt{5}$ times the radius of the projection of the circle.

Hence the locus of Q is a concentric and similar ellipse whose area is 5 times the area of the given ellipse.

13654. (Salutation.)—The triangle ABC has the sides AB, AC bisected in D, E respectively. DE is produced to F, such that EF = DE, and FC is joined. Again, the triangle CEF has its sides CE, CF bisected in G, H respectively, GH is produced to I, such that HI = GH, and IF is joined to form a third independent triangle. The process being continued *ad infinitum*, (1) show that the centroids of the system of triangles are collinear; (2) find the distance apart of the extreme centroids; and (3) find the centre of gravity of the entire system.

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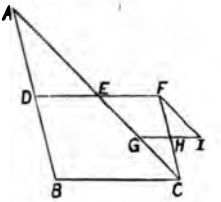
Solution by H. W. Curjel, M.A.; *and* L. E. Reay, B.A.



Let G_1, G_2, G_3, &c., be the centroids of the triangles ABC, CEF, FHI, &c. Since G_1, G_2, G_3, &c., divide the medians through A, C, F, &c., in the ratio 2 : 1 and $\frac{1}{3}-\frac{1}{9}=\frac{1}{3}\times\frac{1}{3}$, therefore G_1, G_2, G_3 are all on the line through G_1 parallel to BC.

Also AG_1, CG_2, FG_3, &c., are clearly parallel.

Therefore $BC = 2G_1G_2 = 4G_2G_3 = \&c.$;

therefore $G_1G_\infty = BC$.

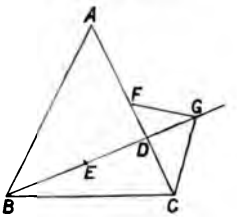
Taking moments about G_1, we see that the distance of the centre of gravity of the system from G_1

$$= BC\,\frac{\frac{1}{2}\cdot\frac{1}{4}+\frac{3}{4}(\frac{1}{4})^2+\frac{7}{8}(\frac{1}{4})^3+\dots}{1+\frac{1}{4}+(\frac{1}{4})^2+\dots} = \tfrac{1}{7}(BC).$$

13947. (P. W. Flood.)—Given one of the segments and the difference of the perpendicular (drawn from one of the equal angles to the opposite side) and other segment of an isosceles triangle, to construct it.

Solution by I. Arnold; F. H. Peachell, B.A.; *and others.*

Analysis.—Let ABC be the isosceles triangle, and BD the perpendicular from one of the equal angles; and AD and BE the given segment and the given difference respectively.



Let DE or DC $= x$; then

$$(BE+x)^2+AD^2 = AB^2 = AC^2 = (AD+x)^2.$$

Then $BE^2+2BE\,.\,x+x^2+AD^2 = AD^2+2AD\,.\,x+x^2$;

or $BE^2+2BE\,.\,x = 2AD\,.\,x$;

consequently x or $DC = \dfrac{\frac{1}{2}BE^2}{AD-BE}$; hence this construction.

Make AD = the given segment and AF = the given difference. Draw a perpendicular through D both ways, and make $DG^2 = \frac{1}{2}AF^2$. Join FG, and from G draw GC perpendicular to FG, cutting AD produced in C. Make DE = DC and EB = AF. Join AB and BC; then is ABC the isosceles triangle required.

5810. (J. L. McKenzie, B.A.)—Given two points P_1, P_2 on a fixed circle, and three points A_1, A_2, A_3 in a straight line. A circle is drawn through either of the points P and any two of the points A, cutting the

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given circle again in H; and a line is drawn joining the remaining points P and A, and cutting the given circle in K. Prove that the six lines HK thus obtained all cut the line ABC in the same point.

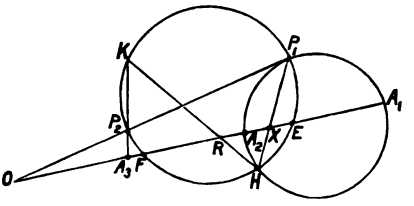
Solution by R. F. Davis, M.A.; *and* F. L. Ward, B.A.

Since

$A_2X \cdot XA_1 = FX \cdot XE$,

we can deduce without difficulty by *componendo*

$EX : FX = EA_1 \cdot EA_2 : FA_1 \cdot FA_2$.



Now $K\{P_2FEH\} = P_1\{P_2FEH\}$,

or $\{A_3FER\} = \{OFEX\}$;

or $A_3F \cdot ER : A_3E \cdot FR = OF \cdot EX : OE \cdot FX$.

Substituting for EX : FX, we find

$RE : FR = -EA_1 \cdot EA_2 \cdot EA_3 \cdot OF : FA_1 \cdot FA_2 \cdot FA_3 \cdot OE$

= a symmetrical quantity.

14089. (D. Edwardes, B.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \sin x \log \sin x \log \cos x \, dx = 2 - \log 2 - \tfrac{1}{8}\pi^2.$$

Solution by G. H. Hardy, B.A.; J. O. Watts; *and many others.*

$$\int_0^{\frac{1}{2}\pi} \sin x \log \sin x \log \cos x \, dx$$

$$= \tfrac{1}{2}\int_0^1 \log t \log (1-t^2)\, dt$$

$$= \frac{1}{2.3^2} + \frac{1}{4.5^2} + \ldots = \left(\frac{1}{2.3} - \frac{1}{3^2}\right) + \left(\frac{1}{4.5} - \frac{1}{5^2}\right) + \ldots$$

$$= \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \ldots\right) - \left(\frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$= 1 - \log 2 - (\tfrac{1}{8}\pi^2 - 1) = 2 - \log 2 - \tfrac{1}{8}\pi^2.$$

Similarly we may find

$$\int_0^{\frac{1}{2}\pi} \sin x \log \cos x \log \frac{\sin}{\cos} \tfrac{1}{2}x \, dx.$$

14192. (Professor Crofton, F.R.S.) — Two points are taken at random in a triangle ABC, the line joining them dividing the triangle into two parts: show that the mean value of the part containing the

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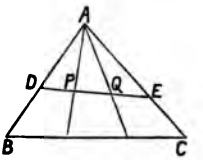
vertex A is $\frac{1}{3}\frac{4}{9}$ of the triangle. Hence it may be shown that the mean value of the *triangular* part is $\frac{4}{9}$ of the whole.

Solution by R. CHARTRES.

Let P and Q be the random points within the triangle.



The mean value of the triangle

$$APQ = \tfrac{2}{3} \text{ of } \tfrac{2}{3} \text{ of } \tfrac{1}{3}\Delta = \tfrac{4}{27}\Delta ;$$

therefore the mean value of the triangle

$$ADE = \tfrac{4}{9}\Delta ;$$

therefore the mean value of the part containing the vertex A $= \frac{1}{3}\Delta\,(\frac{4}{9} + \frac{5}{9} + \frac{4}{9}) = \frac{1}{3}\frac{4}{7}\Delta$.

13827. (F. H. PEACHELL, B.A.)—Find the locus of all points on a rifle range at which the sound of the discharge and the sound of the bullet hitting the target are heard simultaneously. [See BESANT'S *Problems* (page 192, Ed. 1895) and BESANT'S *Solutions* (page 93, Ed. 1890).]

Solution by LIONEL E. REAY, B.A.; Rev. T. MITCHESON, B.A.; *and others.*

A is target, B the rifle, and C the ear.

If velocity of sound is v and of the bullet w, then time from B to C is BC/v, time from B to A is BA/w, time from A to C is AC/v. Then,

from the question, $\quad BC/v = BA/w + AC/v$;

therefore $\quad BC - AC = (v/w)\,BA$;

therefore, if the velocity of the bullet is greater than that of sound, the locus of C is the A branch of the hyperbola with foci A and B and transverse axis equal to (v/w) BA.

If the velocity of the bullet is less than that of sound, there is no point on the range satisfying the conditions.

5840. (A. MARTIN, M.A.)—A catenary is revolved about its axis. If the solid thus formed be cut by a plane, making a given angle with its axis, find the equation to the curve bounding the section.

Solution by Rev. J. CULLEN *and* J. H. DIBB, B.Sc.

The equation of the surface is evidently $x^2 + y^2 = c^2 \cosh^2(z/c)$. Let the given plane be $x \sin\alpha + z\cos\alpha = p$, where α is the angle between the plane and the axis. Putting $\cosh(z/c) = t$ a variable parameter,

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we have $x = (p - c\cos a \cosh^{-1} t)\operatorname{cosec} a,$

$y = \{c^2t^2 - (p - c\cos a \cosh^{-1} t)^2 \operatorname{cosec}^2 a\}^{\frac{1}{2}},$

$z = c\cosh^{-1} t.$

Considered as a plane curve, it takes the form

$$y^2 + (x\cos a - p\sin a)^2 = c^2 \cosh^2 \{(p\cos a + x\sin a)/c\}.$$

14186. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Show that the sum of the *cubes* of any number (>2) of integers in arithmetical progression may always be resolved into *three* factors.

Solution by H. W. CURJEL, M.A.; Professor SANJÁNA, M.A.; *and many others.*

Let $(a+d)$, $(a+2d)$, ..., $(a+nd)$ be the numbers in A.P.

Sum of their cubes

$= na^3 + \frac{1}{2}\{n(n+1)\}\, 3a^2d + \frac{1}{6}\{n(n+1)(2n+1)\}\, 3ad^2 + \frac{1}{4}\{n^2(n+1)^2\}\, d^3$

$= \frac{1}{2}n\{2a + (n+1)d\}\{2a^2 + 4(n+1)ad + n(n+1)d^2\}$

$= \frac{1}{2}\left[n\{2a + (n+1)d\}\{a^2 + 2(n+1)ad + \frac{1}{2}[n(n+1)]d^2\}\right],$

which gives three factors, since one of the first two factors in the numerators is even.

14148. (Rev. T. ROACH, M.A.)—Sum to infinity the series

$$\tfrac{3}{5} - \tfrac{9}{11} + \tfrac{15}{17} - \tfrac{21}{23} + \ldots .$$

Solution by H. W. CURJEL, M.A.; *and the* PROPOSER.

This is an oscillating series, since the limit to which the nth term approaches, when n is infinite, is ± 1.

The sum to $2n$ terms when n is infinite

$= -\frac{2}{5} + \frac{2}{11} - \frac{2}{17} + \frac{2}{23} - \ldots = -2\sum_{1}^{\infty}(-1)^{n+1}/(6n-1)$

$= -\frac{2}{3}(\tan^{-1}1 + \omega\tan^{-1}\omega + \omega^2\tan^{-1}\omega^2)$ [where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$]

$= -\frac{2}{3}\left[\frac{1}{4}\pi + \omega\left\{+\frac{1}{2}i\log(2+\sqrt{3}) - \frac{1}{4}\pi\right\} + \omega^2\left\{-\frac{1}{2}i\log(2+\sqrt{3}) - \frac{1}{4}\pi\right\}\right]$

$= -\frac{2}{3}\left\{\frac{1}{2}\pi + (\omega - \omega^2)\frac{1}{2}i\log(2+\sqrt{3})\right\}$

$= -\frac{2}{3}\left\{\frac{1}{2}\pi - \frac{1}{2}\sqrt{3}\log(2+\sqrt{3})\right\} = -\frac{1}{3}\pi + (1/\sqrt{3})\log(2+\sqrt{3}).$

14068. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—In each of the four triangles formed by intersecting diagonals of a square is inscribed the maximum square. Each of these squares is similarly divided into triangles, in which maximum squares are inscribed; and so on *ad*

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infinitum. Find the ratio of the areas of all the squares to the areas of all the triangles containing them.

Solution by H. W. CURJEL, M.A.

If a is the side of any square, the side of the square inscribed in one of the triangles $= \frac{1}{2}a$. Therefore

$$\text{ratio} = \tfrac{1}{2} : \tfrac{1}{4} = 4 : 9.$$

14076. (Professor THOMAS SAVAGE.) — Show that, n being greater than 2, and s (an integer) a perfect nth power,

$${}^nC_1 x^{n-1} + {}^nC_2 s x^{n-2} + {}^nC_3 s^2 x^{n-3} + \ldots + {}^nC_{n-1} s^{n-2} x + s^{n-1}$$

cannot be a perfect nth power, when x is any positive number (whole or fractional). Also ascertain if the theorem be true for negative values (integral or fractional) of x.

Solution by H. A. WEBB.

Since the equation $X^n + Y^n = Z^n$ cannot be solved in integers if $n > 2$, therefore it cannot be solved in fractions; for, if it could, and we multiplied through by the nth power of the L.C.M. of the denominators of the fractions, we should have solved the equation in integers.

Next, the expression

$$s^{n-1} + ns^{n-2}x + \frac{n(n-1)}{1.2} s^{n-3}x^2 + \ldots + nx^{n-1} = \frac{(s+x)^n - x^n}{s}.$$

If this were equal to a^n, where a is rational, we should have

$$(s+x)^n = x^n + sa^n,$$

which is impossible, if x *is positive,* since s is a perfect nth power.

If x is *negative* and n *even*, a cannot be rational, as before, *unless*

$$s = -2x.$$

If n *odd* and $s > -x$, we have

$$(s+x)^n + (-x)^n = sa^n, \text{ which is impossible.}$$

If n *odd* and $s < -x$, we have

$$(-x)^n = (-x-s)^n + sa^n, \text{ which is impossible.}$$

Hence a *cannot be rational,* for *rational values of* x, unless $x = 0$, or (n *even*) $x = -\frac{1}{2}s$, or (n *odd*) $x = -s$.

14215. (W. C. STANHAM, B.A.)—(1) If P is any point on a rectangular hyperbola which passes through four points A, B, C, D, whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4),

$$\text{P(ABCD)} = \frac{(x_2-x_1)(x_4-x_3) + (y_2-y_1)(y_4-y_3)}{(x_3-x_2)(x_4-x_1) + (y_3-y_2)(y_4-y_1)}.$$

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(2) If A, B, C, D are concyclic, and Q is any point on the circle,

$$Q(ABCD) = \frac{(x_2-x_1)(x_4-x_3)-(y_2-y_1)(y_4-y_3)}{(x_3-x_2)(x_4-x_1)-(y_3-y_2)(y_4-y_1)} = \frac{(x_2-x_1)(y_4-y_3)+(y_2-y_1)(x_4-x_3)}{(x_3-x_2)(y_4-y_1)+(y_3-y_2)(x_4-x_1)}.$$

The axes of coordinates are considered to be rectangular in both (1) and (2), and E(ABCD) to represent the ratio

$$(\sin AEB \,.\, \sin CED) / (\sin BEC \,.\, \sin AED).$$

Solution by the PROPOSER.

Let PA, PB, PC, and PD meet the axis of x at D_1, D_2, D_3, and D_4. Then $P(ABCD) \equiv \lambda = D_1D_2 \,.\, D_3D_4 / D_2D_3 \,.\, D_1D_4$.

Therefore, if P is (x', y'), we have

$$f(x', y', \lambda, x_1, y_1, x_2, y_2, \&c.) = 0,$$

where f denotes a function of the second degree in x' and y'. This will give the equation of the locus of P if λ is chosen so that the sum of the coefficients of x'^2 and y'^2 is zero. This gives the value of P(ABCD) in the Question. Similarly, for a circle, the coefficients of x'^2 and y'^2 are equal, and the coefficient of $x'y'$ is zero. Whence the value of Q(ABCD), and the identical relation between x_1, y_1, x_2, y_2, &c.

14084. (G. HALE PUCKLE, M.A.)—With origin transferred to a real focus and with parallel axes, the equation $\phi(xy) = 0$ becomes (Quest. 13535, No. 437)

$$\rho^4(ax^2 + 2hxy + by^2) - 2\rho^2\Delta^{\frac{1}{2}}(\alpha x + \beta y) + \Delta = 0,$$

and the equation to the directrix becomes

$$\rho^2(\alpha x + \beta y) - \Delta^{\frac{1}{2}} = 0.$$

Solution by the PROPOSER.

If in the equations $\phi(xy) = 0$ and $\alpha x + \beta y + \gamma = 0$, representing a conic and its directrix, the origin is transferred to the focus $(x'y')$, then (Quests. 13535, 13572, No. 437)

$$\rho^2(ax' + hy' + g) + \alpha\Delta^{\frac{1}{2}} = 0, \quad \rho^2(hx' + by' + f) + \beta\Delta^{\frac{1}{2}} = 0,$$
$$\rho^2(ax' + \beta y' + \gamma) + \Delta^{\frac{1}{2}} = 0, \quad \phi(x'y') = \Delta/\rho^4, \ \&c.$$

The axis of real foci is parallel to $\beta x - \alpha y = 0$.

14098. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Show that the sum of any number (>2) of successive cubes can always be resolved into *three* (algebraic) factors.

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E.g.—Resolve the sum of the 8 million cubes from 1,000,001 to 9,000,000 into its prime factors.

Solution by the PROPOSER.

Let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$; $S_m = 1^3 + 2^3 + 3^3 + \ldots + m^3$.

Let $S_{m,n} = (m+1)^3 + (m+2)^3 + \ldots + n^3$; $(n > m)$.

Then $S_{m,n} = S_n - S_m = \left\{\frac{1}{2}n(n+1)\right\}^2 - \left\{\frac{1}{2}m(m+1)\right\}^2$

$= \frac{1}{4}\left\{(n^2+n) - (m^2+m)\right\}\left\{(n^2+n) + (m^2+m)\right\}$

$= \frac{1}{4}(n-m).(n+m+1).(n^2+n+m^2+m).$

Ex.—Let $m = 1{,}000{,}000 = 10^6$; $n = 9{,}000{,}000 = 9 \times 10^6$;

therefore $S_{m,n} = \frac{1}{4} \times (8.10^6)(10^7+1)(8{,}200{,}001 \times 10^7)$

$= 2 \times 10^{13} \times (11 \times 909{,}091) \times (17 \times 19 \times 53 \times 479).$

14086. (C. N. CAMA, M.A.)—Show that the asymptotes of the curve whose r-p equation (the so-called "pedal" equation) is

$$(a - 3p^4)r^4 + 4r^3p^3 - 6r^2p^2 + 4rp = 1$$

all touch a circle of radius $(\frac{1}{3}a)^{\frac{1}{2}}$.

Solution by H. A. WEBB; H. W. CURJEL, M.A.; *and many others.*

At the point of contact of an asymptote with the curve, r is infinite.

Now, whenever r is infinite, $p = \pm(\frac{1}{3}a)^{\frac{1}{2}}$.

Hence every asymptote is distant $(\frac{1}{3}a)^{\frac{1}{2}}$ from the origin, and touches the circle whose centre is the origin and radius $(\frac{1}{3}a)^{\frac{1}{2}}$.

The equation of the curve gives $p \pm (a - 2p^4)^{\frac{1}{2}} = 1/r$,

showing that, when $p = \pm(\frac{1}{3}a)^{\frac{1}{2}}$, $r = \infty$,

and no asymptote cuts the curve at a finite distance from the origin.

The above theorem is true also for the general curve

$$(a - 3p^4)r^4 + (A + Bp + Cp^2 + Dp^3)r^3 + (E + Fp + Gp^2)^2 + (H + Kp)r = L.$$

14143. (Professor UMES CHANDRA GHOSH).—Prove that

$$\int_0^\infty \frac{\cot x - 2^a \cot 2^a . x}{x}\, dx = \tfrac{1}{2}\pi(2^a - 1),$$

where a is any positive integer.

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Solutions (1) *by* Rev. J. CULLEN; (2) *by* Professor SANJÁNA.

(1) We have

$$\sin x/(1+2m\cos x+m^2) = \sin x - m\sin 2x + m^2\sin 3x - \dots;$$

hence, dividing across by x and putting $m=1$, we get

$$\int_0^\infty \tan x\,\frac{dx}{x} = \tfrac{1}{2}\pi, \quad \text{since} \quad \int_0^\infty \sin rx\,\frac{dx}{x} = \tfrac{1}{2}\pi;$$

also

$$\int_0^\infty (\cot x - \tan x)\,\frac{dx}{x} = 2\int_0^\infty \cot 2x\,\frac{dx}{x};$$

therefore

$$\int_0^\infty \cot x\,\frac{dx}{x} = -\tfrac{1}{2}\pi;$$

whence the result, since, in general,

$$\int_0^\infty (\cot x - p\cot qx)\,\frac{dx}{x} = \tfrac{1}{2}\pi\,(p-1).$$

(2) Assuming that $\int_0^\infty \frac{\tan ax}{x}\,dx = \tfrac{1}{2}\pi$, and remembering that

$$\tan x + 2\tan 2x + 4\tan 4x \dots + 2^{a-1}\tan 2^{a-1}x = \cot x - 2^a\cot 2^a x,$$

we get the integral $= \tfrac{1}{2}\pi\,(1+2+5\dots+2^{a-1}) = \tfrac{1}{2}\pi\,(a^a-1)$.

A satisfactory proof of the first equation has yet to be given. [See below.]

Note on Quests. 14143 *and* 14171 *by* H. W. CURJEL, M.A.

(14143.) If it is assumed that $\int_0^\infty \frac{\tan x}{x}\,dx$ is convergent, we get

$$\int_0^\infty \frac{\tan x}{x}\,dx = \int_0^\infty \frac{\tan \pi x}{x}\,dx = \int_{-\infty}^0 \frac{\tan \pi x}{x}\,dx = \tfrac{1}{2}\int_{-\infty}^\infty \frac{\tan \pi x}{x}\,dx$$

$$= \tfrac{1}{2}\int_0^1 \tan \pi x \sum_{-\infty}^{\infty} \frac{\pi}{\pi x + r\pi}\,dx = \tfrac{1}{2}\int_0^1 \tan \pi x \,.\, \pi\cot \pi x\,dx = \tfrac{1}{2}\pi;$$

therefore given integral

$$= \int_0^\infty \frac{(\cot x - 2\cot 2x) + (2\cot 2x - 2^2\cot 2^2x) + \dots}{x}\,dx$$

$$= \int_0^\infty \left\{ \frac{\tan x}{x}\,dx + \frac{2\tan 2x}{2x}\,d(2x) + \dots \right\} = (2^a-1)\int_0^\infty \frac{\tan x}{x}\,dx$$

$$= (2^a-1)\,\tfrac{1}{2}\pi.$$

This result, however, depends on the assumption that $\int_0^\infty \frac{\tan x}{x}\,dx$ is convergent, which is not true, as may be seen as follows:—One of the necessary conditions for the convergency of $\int_0^\infty \phi(x)\,dx$ is that $\int_n^{n+m} \phi(x)\,dx$ may be made less than any assignable quantity, however small, by

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taking n large enough. This condition is not satisfied in the case of $\int_0^\infty \frac{\tan x}{x}\,dx$, as $\int_n^{n+m} \frac{\tan x}{x}\,dx$ is infinite if $m+n = k\pi + \frac{1}{2}\pi$. However, it is easy to prove that, m being integral, the limit of $\int_0^{m\pi+\theta} \frac{\tan x}{x}\,dx$, when m is infinite, $= \frac{1}{2}\pi$ if θ differs by a finite quantity from $\frac{1}{2}\pi$, π being $> \theta > 0$. If, however, θ is a function $\psi(m)$ of m such that $\theta = \frac{1}{2}\pi$ when m is infinite, by properly choosing the function ψ we can make the limit of $\int_0^{m\pi+\theta} \frac{\tan x}{x}\,dx$, when m is infinite, equal to any quantity we please between $\frac{1}{2}\pi$ and infinity.

(14171.) The results $\int_0^\infty \cos rx\,dx = 0$, $\int_0^\infty \sin rx\,dx = 1/r$ are incorrect, as the first two integrals in the Question are convergent only if a is positive, and are clearly not so when a is zero. It is possible to show, by an equally sound (or rather equally unsound) method, $\cos\infty$ and $\sin\infty$ are equal to any quantities we please between -1 and $+1$. The error in the method is similar to that in any method which assigns a definite sum to an infinite oscillating series, such as assigning to $1-1+1-1+\ldots$ ad inf. the value $\frac{1}{2}$, or assigning to it the value 0.

14107. (Professor Neuberg.)—Résoudre le système

$$\tan(y+z) = a\tan x, \quad \tan(z+x) = b\tan y, \quad \tan(x+y) = c\tan z.$$

Solution by V. R. Thyagarayaiyar, M.A.; *and* L. E. Reay, B.A.

The equations may be written

$$\left.\begin{aligned} -a\tan x + \tan y + \tan z + a\lambda &= 0 \\ \tan x - b\tan y + \tan z + b\lambda &= 0 \\ \tan x + \tan y - c\tan z + c\lambda &= 0 \end{aligned}\right\},$$

where $$\lambda = \tan x \tan y \tan z;$$

therefore $$\tan x \begin{vmatrix} -a & 1 & 1 \\ 1 & -b & 1 \\ 1 & 1 & -c \end{vmatrix} = -\lambda \begin{vmatrix} a & 1 & 1 \\ b & -b & 1 \\ c & 1 & -c \end{vmatrix},$$

or $$\tan y \tan z = \frac{abc - \Sigma a - 2}{abc + 2bc + b + c - a},$$

with two similar equations; therefore

$$\tan x = \frac{(abc-\Sigma a-2)^{\frac{1}{2}}(abc+2bc+b+c-a)^{\frac{1}{2}}}{(abc+2ac+c+a-b)^{\frac{1}{2}}(abc+2ab+a+b-c)^{\frac{1}{2}}}.$$

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14236. (C. E. Hillyer, M.A.)—O is the circumcentre and I the incentre of a triangle ABC; AO, BO, CO meet the circumcircle again in a, b, c respectively; X, Y, Z are the points of contact of the incircle with the sides, and Xδ, Yϵ, Zζ the perpendiculars of the triangle XYZ. Prove that $a\delta$, $b\epsilon$, $c\zeta$ are concurrent in I.

Solution by R. F. Davis, M.A.; J. A. Third, M.A.; *and others.*

Let AI meet the circumcircle in E and YZ in M. Then

$$\angle IX\delta = OEA$$
$$= OAE;$$

therefore

$$\delta M : IX = aE : Aa.$$

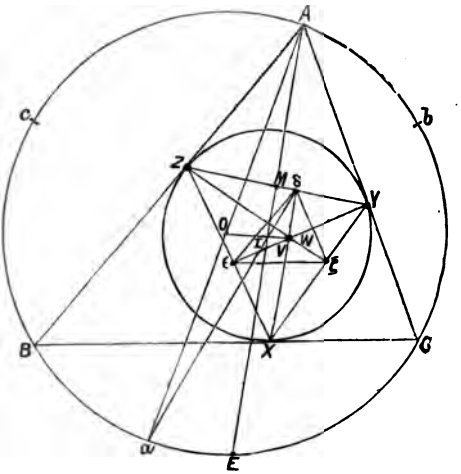
But

$$IX : IM = IY : IM$$
$$= 2R : EC$$
$$= Aa : EI.$$

Hence

$$\delta M : IM = aE : EI,$$

or $a\delta$ passes through I.



[The Proposer solves the Question as follows:—If aI meet YZ in δ,

$$I\delta : Ia = IM : IE = IM.IA : IE.IA$$
$$= IY^2 : IE.IA = r^2 : 2R.r = \tfrac{1}{2}r : R.$$

Similarly, if bI meet ZX in ϵ, $I\epsilon : Ib = \frac{1}{2}r : R$;

therefore δE is parallel to ab, and $\delta E : ab = \frac{1}{2}r : R$.

But ab is equal and parallel to AB; therefore δ, ϵ are the feet of the perpendiculars from X, Y on YZ, ZX; and similarly for ζ, which proves the proposition.]

14090. (G. H. Hardy, B.A.)—Evaluate

$$\int_0^\infty \tan^{-1}(p \operatorname{sech} u)\, du \quad \text{and} \quad \int_0^\infty \tan^{-1}\left\{\tfrac{1}{2}\left(a - \frac{1}{a}\right)\operatorname{sech}^2 u\right\} du.$$

Solution by H. A. Webb.

(1) $$I = \int_0^\infty \tan^{-1}(p \operatorname{sech} u)\, du.$$

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Put $\operatorname{sech} u = \cos x, \quad x = gdu, \quad du = \sec x\,dx.$

$$\therefore I = \int_0^{\frac{1}{2}\pi} \tan^{-1}(p\cos x).\sec x\,dx = \int_0^{\frac{1}{2}\pi} \left\{p - \tfrac{1}{2}p^3\cos^2 x + \tfrac{1}{3}p^5\cos^4 x - \ldots\right\} dx$$

$$= \left\{p - \tfrac{1}{2}p.\tfrac{1}{2} + \tfrac{1}{3}p^5.\frac{1.3}{2.4} - \ldots\right\}\tfrac{1}{2}\pi = \tfrac{1}{2}\pi\int_0^p \frac{dp}{\sqrt{(1+p^2)}} = \tfrac{1}{2}\pi\sinh^{-1}p.$$

(2) $$I = \int_0^\infty \tan^{-1}\left\{\tfrac{1}{2}\left(a - \frac{1}{a}\right)\operatorname{sech}^2 u\right\} du$$

$$= \int_0^{\frac{1}{2}\pi} \sec x \tan^{-1}\left\{\tfrac{1}{2}\left(a - \frac{1}{a}\right)\cos^2 x\right\} dx,$$

from same substitution as before,

$$= \int_0^{\frac{1}{2}\pi} \left\{\tfrac{1}{2}\left(a - \frac{1}{a}\right)\cos x - \frac{1}{3}\cdot\frac{1}{2^3}\left(a - \frac{1}{a}\right)^3\cos^5 x + \ldots\right\} dx;$$

$$\therefore I = \frac{1}{2}\left(a - \frac{1}{a}\right) - \frac{1}{2^3}\cdot\left(a - \frac{1}{a}\right)^3\cdot\frac{2.4}{3.5} + \frac{1}{2^5}\cdot\left(a - \frac{1}{a}\right)^5\cdot\frac{2.4.6.8}{3.5.7.9}\ldots;$$

therefore I is the real part of

$$-i\left\{\sin^{-1}\left[\frac{i}{2}\left(a - \frac{1}{a}\right)\right]^{\frac{1}{2}}\right\}^2.$$

[The Proposer solves the Question as follows:—Let

$$P = \int_0^\infty \log(1 + p\operatorname{sech} u)\,du \qquad (p < 1), \qquad \frac{dP}{dp} = \int_0^\infty \frac{du}{\cosh u + p},$$

and putting $(\cosh u + p)(\cos x - p) = 1 - p^2,$

this becomes $$\frac{dP}{dp} = \frac{\cos^{-1} p}{\sqrt{(1-p^2)}},$$

whence $$P = \tfrac{1}{2}\left\{(\tfrac{1}{2}\pi)^2 - (\cos^{-1}p)^2\right\} \ldots\ldots\ldots\ldots (1).$$

Now it is easy to justify our putting $p = iq$, and then, equating real and imaginary parts, we find

$$\int_0^\infty \tan^{-1}(q\operatorname{sech} u)\,du = \tfrac{1}{2}\pi\sinh^{-1}q \ldots\ldots\ldots\ldots (2),$$

$$\int_0^\infty \log(1 + q^2\operatorname{sech}^2 u)\,du = \left\{\sinh^{-1}q\right\}^2 \ldots\ldots\ldots\ldots (3).$$

In (3) put $q = \sqrt{(ik)}$, as is permissible, and equate imaginary parts; we get $$\int_0^\infty \tan^{-1}(k\operatorname{sech}^2 u)\,du = \mathrm{R}\left[-i\left\{\sinh^{-1}\sqrt{(ik)}\right\}^2\right] \ldots\ldots\ldots (4),$$

or $$= \mathrm{R}\left[-i\left\{\sin^{-1}\sqrt{(ik)}\right\}^2\right],$$

agreeing with Mr. Webb's solution.

If, moreover, we put $k = \frac{1}{2}(a - 1/a)$, this may be expressed very simply; for

$$\sinh^{-1}\sqrt{(ik)} = \log\left[\sqrt{(ik)} + \sqrt{(1+ik)}\right] = \log(u + iv),$$

where $$u = (1/\sqrt{2})\left[\sqrt{k} + \sqrt{\left\{\sqrt{(1+k^2)} + 1\right\}}\right],$$

$$v = (1/\sqrt{2})\left[\sqrt{k} + \sqrt{\left\{\sqrt{(1+k^2)} - 1\right\}}\right];$$

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and, if $a = e^{\theta}$ $k = \sinh\theta$, $(a > 1)$,

$$u = \surd(\cosh\tfrac{1}{2}\theta)\left[\surd(\sinh\tfrac{1}{2}\theta) + \surd(\cosh\tfrac{1}{2}\theta)\right],$$

$$v = \surd(\sinh\tfrac{1}{2}\theta)\left[\surd(\sinh\tfrac{1}{2}\theta) + \surd(\cosh\tfrac{1}{2}\theta)\right],$$

$$\log(u^2+v^2) = \log\left\{e^{\theta} + \surd(e^{2\theta}-1)\right\} = \cosh^{-1}a$$

$$\tan^{-1}v/u = \tfrac{1}{2}\cos^{-1}(e^{-\theta}) = \tfrac{1}{2}\cos^{-1}(1/a),$$

and $$\int_0^{\infty}\tan^{-1}\left\{\tfrac{1}{2}(a-1/a)\operatorname{sech}^2 u\right\}du = \log(u^2+v^2)\tan^{-1}(v/u)$$

$$= \tfrac{1}{2}\cosh^{-1}a\cos^{-1}(1/a),$$

at any rate when $a > 1$, $a - 1/a < 1$.]

14079. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Exhibit the product of the two series

$1-8+27-64+125-216+\ldots$ and $1-16+81-256+625-1296+\ldots$,

each to n terms, as a function of the roots of a quadratic equation.

Solution by Lt.-Col. Allan Cunningham, R.E.

Two cases arise, according as the number of terms (n) is *odd* or *even*.

Let $S_3 = 1^3-2^3+3^3-\ldots+(2m+1)^3$; $S_4 = 1^4-2^4+3^4-\ldots+(2m+1)^4$;

$S_3' = 1^3-2^3+3^3-\ldots-(2m)^3$; $S_4' = 1^4-2^4+3^4-\ldots-(2m)^4$.

Then $S_3 = \left\{1^3+2^3+3^3+\ldots+(2m+1)^3\right\} - 2\left\{2^3+4^3+\ldots+(2m)^3\right\}$;

$S' = \left\{1^3+2^3+3^3+\ldots+(2m)^3\right\} - 2\left\{2^3+4^3+\ldots+(2m)^3\right\}$.

Therefore $S_3 = \left\{\tfrac{1}{2}(2m+1)(2m+2)\right\}^2 - 2^4\left\{\tfrac{1}{2}m(m+1)\right\}^2$;

$S_3' = \left\{\tfrac{1}{2}.2m(2m+1)\right\}^2 - 2^4\left\{\tfrac{1}{2}m(m+1)\right\}^2$.

Also $S_4 = \left\{1^4+2^4+3^4+\ldots+(2m+1)^4\right\} - 2\left\{2^4+4^4+\ldots+(2m)^4\right\}$;

$S_4' = \left\{1^4+2^4+3^4+\ldots+(2m)^4\right\} - 2\left\{2^4+4^4+\ldots+(2m)^4\right\}$;

and the product of the two series is $S_3.S_4$ or $S_3'.S_4'$, according as $n = 2m+1$ or $n = 2m$.

Lastly, S_3, S_4, or S_3', S_4', are the roots of the quadratic

$$x^2-(S_3+S_4)x+S_3S_4 = 0 \quad \text{or} \quad x^2-(S_3'+S_4')x+S_3'S_4' = 0,$$

whose absolute term = the required product.

[The summations for S_3, S'_3 are taken from Chrystal's *Algebra*, Part I., Chap. xx., Art. 7; those of the parts S_4, S'_4 are also given there.]

14172. (Professor Thomas Savage.)—ABC is an isosceles triangle; DE, a parallel to BC, intersects AB, AC in D, E respectively. If DE, DB, EC are all equal, show that BC is not one of the equal sides of the triangle.

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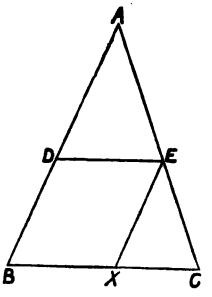
Solutions by E. WALTER TODD; Prof. LANGHORNE ORCHARD, M.A., B.Sc.; *and many others.*



Of course the triangle is assumed not equilateral.

Draw EX parallel to BA. Then, since DX is a parallelogram, $EX = BD$, (by I. 34)

$= EC$; (by hyp.)

therefore $\angle EXC = C$. (by I. 5)

But $\angle EXC = B$; (by I. 29)

therefore $\angle B = C$;

therefore BC is base of isosceles triangle ABC, and therefore not one of equal sides.

The theorem is true if BD = EC without each being equal to DE.

14177. (B. N. CAMA, M.A.)—Prove that the locus of the intersection of tangents, including a given angle β, to the spiral $r = ae^{\theta \cot \alpha}$ is the copolar spiral

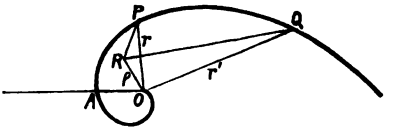
$$\frac{r \sin \psi}{\sin \alpha} = ae^{(\theta+\psi-\alpha)\cot\alpha},$$

where ψ is given by $\cot \psi = \dfrac{k - \cos \beta}{\sin \beta}$

and ak is the vertical distance of a point on the original curve whose vectorial angle is β. Show also that the locus touches the circle round the triangle formed by the tangents and the chord of contact.

Solution by Rev. J. CULLEN; G. BIRTWISTLE, B.Sc.; *and the* PROPOSER.

Let $\angle RQO = RPO = \gamma$ any angle, ROA = ϕ, and QOA − POA = $\theta' - \theta = \beta$.



Then $\dfrac{r}{\rho} = \dfrac{\sin(\gamma + \theta - \phi)}{\sin \gamma}$,

$\dfrac{r'}{\rho} = \dfrac{\sin(\gamma + \theta' - \phi)}{\sin \gamma}$.

Therefore $r^2 - 2rr' \cos \beta + r'^2 = \rho^2 \sin^2 \beta \operatorname{cosec}^2 \gamma$;

also $r'/r = k = \sin(\gamma + \beta + \theta - \phi) / \sin(\gamma + \theta - \phi)$;

therefore $\cot(\gamma + \theta - \phi) = (k - \cos \beta)/\sin \beta = \cot \psi$,

if $\psi = \gamma + \theta - \phi$, and $r \sin \gamma = \rho \sin \psi$;

therefore $\rho \dfrac{\sin \psi}{\sin \gamma} = ae^{(\phi+\psi-\gamma)\cot\alpha}$, since $r = ae^{\theta \cot \alpha}$.

But, if $\gamma = \alpha$, QR and PR become tangents; hence the first result. In this case also the tangent at R to the circle PQR makes an angle α with

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OR; therefore the locus of R and the circle have a common tangent; hence they touch.

[Some interesting results follow by giving γ different values, *e.g.*, $\gamma = \psi$ or $\frac{1}{2}\pi - a$, &c.]

14139. (Professor Neuberg.)—Eliminer x, y, z entre des équations

$$\cos x \sin y \sin z = a, \qquad \cos y \sin z \sin x = b,$$
$$\cos z \sin x \sin y = c, \qquad \cos x \cos y \cos z = d.$$

Solution by Rev. T. Roach, M.A.; *and* F. H. Peachell, B.A.

$$\tan^2 x = \frac{bc}{ad}; \quad \text{therefore} \quad \cos^2 x = \frac{ad}{ad+bc}.$$

Similarly with y, z; therefore

$$d^2 = \frac{ad}{ad+bc} \cdot \frac{bd}{ac+bd} \cdot \frac{cd}{ab+cd};$$

therefore $$abcd = (ad+bc)(ab+cd)(ac+bd).$$

14180. (W. C. Stanham, B.A.)—If O is a fixed point, and PQ is any chord of a conic drawn through one of its foci (S), and if OP, OQ meet the conic again at P′ and Q′, P′Q′ passes through a fixed point (Σ), and, conversely, if P′Q′ is any chord through Σ, and if OP′, OQ′ meet the conic again at P and Q, PQ passes through S.

Solution by H. W. Curjel, M.A.; *and* G. D. Muggeridge.

This theorem is true if S is any point.

If OS cuts the polar of O in R, then OR is divided harmonically by PQ, P′Q′, *i.e.*, in S and Σ; therefore, if S is a fixed point, Σ is a fixed point, and, conversely, if Σ is a fixed point, S is a fixed point.

[The Proposer observes that this is the polar form of Quest. 14161 and its converse.]

14137. (Professor Crofton, F.R.S.)—Prove that $(3n)^{4n} - 1$ is divisible by $24n+1$, if the latter is a prime number.

Solution by Lt.-Col. Allan Cunningham, R.E.; *and* H. W. Curjel, M.A.

Let $24n+1 = p$ (a prime).

Then $24n+1 \equiv 0 \pmod{p}$ and $2^3 . 3n \equiv -1 \pmod{p}$;

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therefore $(2^3)^{\frac{1}{2}(p-1)} . (3n)^{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \pmod{p}$;

therefore $2^{\frac{1}{2}(p-1)} . (3n)^{4n} \equiv (-1)^{4n} \equiv 1 \pmod{p}$.

But $2^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$, because $p = 8\varpi + 1$; therefore

$$(3n)^{4n} - 1 \equiv 0 \pmod{p}.$$

[*Similarly*, it may be shown that $(mn)^{2^{m-1}n} - 1$ is divisible by $(2^{2m}mn + 1)$ if the latter is *prime*; *e.g.*, $(5n)^{16n} - 1$ is divisible by $(160n + 1)$ when the latter is prime.]

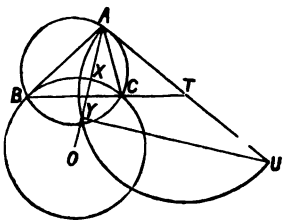
14237. (R. F. Davis, M.A.)—The tangent at A to the circumcircle of the triangle ABC meets BC in T and is produced to U so that AT = TU. Prove that the polar of A with regard to any circle through B, C passes through U.

Solutions (1) *by* J. A. Third, M.A., *and many others*; (2) *by* G. D. Muggeridge *and* Professor A. Droz-Farny.

(1) Let O be the centre of any circle through B, C; and let OA meet this circle in X and the circle having AU as diameter in Y. Circle T is obviously orthogonal to all circles through B, C; therefore OX = tangent from O to circle T; therefore

$$OX^2 = OY . OA;$$

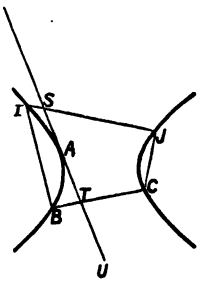
and AYU is a right angle; therefore YU is the polar of A with regard to circle O.



(2) Consider a system of conics passing through the four points B, C, I, J. Of these conics one passes through a given point A. Draw the tangent at A to this conic, meeting IJ, BC in S and T. Take U the fourth harmonic to S, A, T; then A, U are the double points of the involution on the line AT formed by the points of intersection of this line and the given system of conics.

Hence A, U are conjugate points for all conics passing through B, C, I, J.

In the particular case when I, J are the circular points, all the conics are circles, and, S being at an infinite distance, T bisects AU.



14122. (The late "Lewis Carroll.")—It is given that (1), if C is true, then, if A is true, B is not true; and (2), if A is true, B is true. Can C be true? What difference in meaning, if any, exists between the

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following propositions?—(1) A, B, C cannot be all true at once; (2) if C and A are true, B is not true; (3) if C is true, then, if A is true, B is not true.

Solution by H. W. CURJEL, M.A.

Using the notation of my solutions to Quests. 13234 and 13288 (Vol. LXVI.), (1) gives CAB = 0, (2) gives AB = A; therefore CA = 0; therefore either A or C is not true; therefore C may be true if A is not true.

The statements at the end of the question are equivalent, for

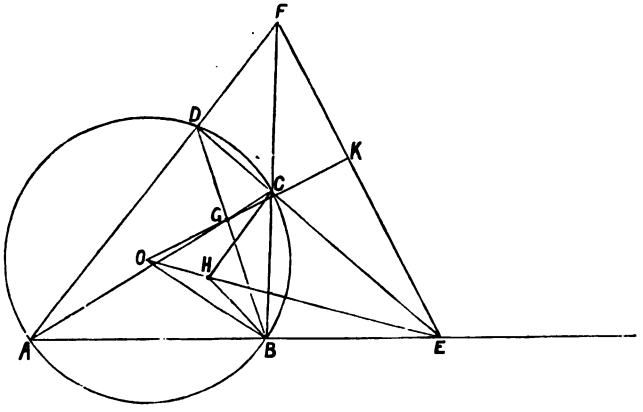
$$ABC = AC.B = A.C.B.$$

14176. (R. F. DAVIS, M.A.)—ABCD is a concyclic quadrilateral, and E, F, G are the intersections (AB, DC), (AD, BC), and (AC, BD) respectively. Show that the focus of the parabola escribed to ABCD is the foot of the perpendicular from G on EF.

Solution by H. W. CURJEL, M.A.; *and* Professor SANJÁNA, M.A.

Let O be the centre of circle ABCD, and let H be the inverse point of E with respect to circle ABCD (H is cut of DE and FG).

$$\angle BHC = OBA + OCD = \tfrac{1}{2}\{AOD + BOC\} = CGB;$$



therefore B, C, G, B are concyclic, and circle BCGH is inverse of circle BEC, which therefore passes through the inverse of G, *i.e.*, through K, the foot of the perpendicular from G on EF, since EF is the polar of G. Similarly, circle FCD passes through K; therefore K is the focus of the parabola escribed to ABCD.

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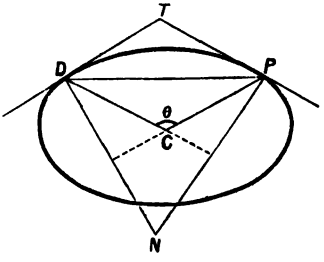
14185. (Rev. T. MITCHESON, B.A.)—If R is the point of intersection of the normals drawn from the ends of semi-conjugate diameters of an ellipse, show that the distance of R from the centre is

$$x_1y_1a^{-3}b^{-3}(a^2-b^2)\left\{b^2(bx_1-ay_1)^2+a^2(bx_1+ay_1)^2\right\}^{\frac{1}{2}},$$

(x_1y_1) being the extremity of one of the diameters.

Solution by G. BIRTWISTLE, B.Sc.; R. TUCKER, M.A.; *and others.*



CP and CD are semi-conjugates, PN and DN normals intersecting in N.

Thus, N is orthocentre of triangle CPD, and thus, by trigonometry,

$$CN = PD\cot\theta.$$

P is (x_1y_1), and therefore D is

$$(-ay_1/b,\ bx_1/a);$$

therefore

$$PD^2 = (x_1+ay_1/b)^2+(y_1-bx_1/a)^2$$
$$= 1/a^2b^2\left\{(bx_1+ay_1)^2a^2+(bx_1-ay_1)b^2\right\},$$

and $CP.CD.\sin\theta = ab$;

therefore $\cot\theta = 1/ab\sqrt{(CP^2.CD^2-a^2b^2)}$,

$$CP^2.CD^2-a^2b^2 = (x_1^2+y_1^2)(a^2y_1^2/b^2+b^2x_1^2/a^2)-a^2b^2$$
$$= \left\{(x_1^2/a^2+y_1^2/b^2)^2-1\right\}a^2b^2+x_1^2y_1^2(a^2-b^2)^2/a^2b^2$$
$$= \left\{x_1^2y_1^2(a^2-b^2)^2\right\}/a^2b^2,$$

since $x_1^2/a^2+y_1^2/b^2 = 1$;

$$\therefore\ CN = 1/ab\left\{(bx_1+ay_1)^2a^2+(bx_1-ay_1)b^2\right\}^{\frac{1}{2}}.1/ab.\left\{x_1y_1(a^2-b^2)\right\}/ab$$
$$= x_1y_1a^{-3}b^{-3}(a^2-b^2)\left\{a^2(bx_1+ay_1)^2+b^2(bx_1-ay_1)^2\right\}^{\frac{1}{2}}.$$

13951. (R. KNOWLES, B.A.)—If the diagonals of a polygon, having an even number of sides, inscribed in a conic, concur in a point G, prove that (1) the diagonals of the corresponding circumscribed polygon also concur in G; (2) hence the diagonals of a polygon, having an even number of sides, circumscribed to a conic are concurrent.

Solution by H. W. CURJEL, M.A.

Project the points of intersection with the conic of the polar of G into the circular points at infinity; the conic then becomes a circle with centre G, and the first part of the Question becomes obvious; hence it is also clear that to every conic there may be circumscribed an infinite number of polygons of a given number of sides whose diagonals are concurrent. (2) However, is not true when the number of sides is greater than 6; since, having drawn a circumscribed polygon whose diagonals are concurrent, by displacing a side we cause two of the diagonals no longer to pass through the former point of concurrence, whereas the intersection of the remaining diagonals remains fixed.

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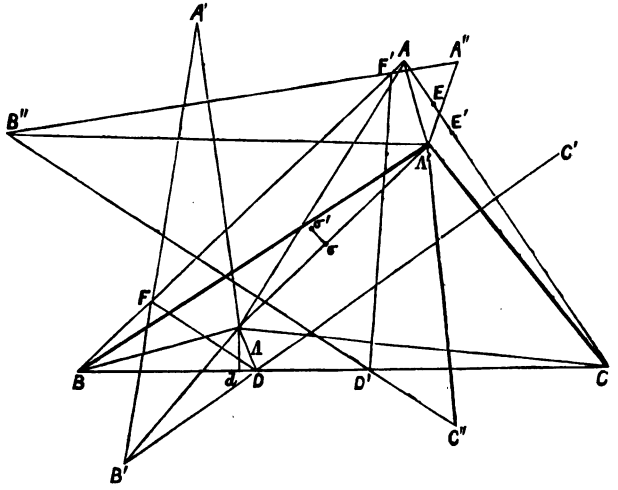
14182. (Rev. J. Cullen.)—A triangle A′B′C′ is obtained by revolving a given triangle ABC on a point Λ in its plane through an angle θ; A″B″C″ is similarly obtained by revolving ABC on Λ′, the isogonal conjugate of Λ in an opposite direction $(-\theta)$. Prove that (1) the six intersections of the homologous sides of A′B′C′ and A″B″C″ with those of ABC lie on a circle C_θ. (2) As θ varies the locus of the centre of C_θ is the perpendicular to ΛΛ′ through its mid-point. (3) The envelope of C_θ is the inscribed conic whose foci are Λ and Λ′. (4) If Λ lies on the circum-circle of ABC, C_θ becomes a right line whose envelope is a parabola.

Solution by G. N. Bates.

Let the intersections of the homologous sides of A′B′C′, A″B″C″ with those of ABC be DD′, EE′, FF′.



Then BB′, DΛF are concyclic, as are also BB″, F′Λ′D′.

Therefore ∠ BFD = supplement of BB′D = supplement of BB″F′ = BD′F′; therefore triangles BFD, BD′F′ are similar; therefore BD.BD′ = BF.BF′, and similarly AF.AF′ = AE.AE′ and CD.CD′ = CE.CE′;

therefore by a known theorem DD′, EE′, FF′ are concyclic ..... (1).

Now draw Λ*d*, Λ*e*, Λ*f* perpendicular to BC, &c. Then triangles DEF, *def* are similar, Λ is their centre of similitude, and ∠ DΛ*d* = ½θ; therefore

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if ρ and σ be the radius and centre of circle *def*, and ρ' and σ' the radius and centre of DEF, $\rho' = \rho/\cos\frac{1}{2}\theta$, and $\sigma\sigma'$ is perpendicular to $A\sigma$;

therefore locus of σ' is the perpendicular bisector of AA' ..... (2).

Now let e be the eccentricity of inscribed conic whose foci are A and A'.

Then $A\sigma' = A\sigma/\cos\frac{1}{2}\theta = e\rho/\cos\frac{1}{2}\theta = e\rho'$;

therefore the envelope of the circles C_θ is this conic ..... (3).

If A lies on the circumcircle, A' is at infinity, *def* becomes the pedal line of A, and DEF becomes a right line enveloping a parabola focus A having *def* for its tangent at the vertex ..... (4).

[The PROPOSER adds: "It is easy to prove that where C_θ cuts the sides of ABC two triangles may be inscribed similar to the pedal triangles of A and A', the modulus of similarity being $\sec\frac{1}{2}\theta$, so that, if A be a B.-point, C_θ becomes a TUCKER-circle, while, if it be taken as the circumcentre, we have a generalization of the N.P.-circle, since C is then the common circumcircle of two triangles inscribed to ABC, and similar to the medial and pedal triangles.]

5669 & 5804. (Rev. W. A. WHITWORTH, M.A.)—(5669.) A man drinks (in random order) n glasses of wine and n glasses of water (all equal). Show that the odds are n to 1 against his never having drunk, throughout the operation, more wine than water.

(5804.) A man plays $3n+2$ times, for an even stake, winning $2n+1$ times and losing $n+1$ times. Show that the chance that throughout the play he is never either worse off than at the beginning, or better off than at the end, is $n/(4n+6)$.

Solutions by H. FORTEY.

(5669.) Let $u_{x,y}$ be the number of ways of drinking x glasses of water and y glasses of wine, subject to the condition that x is never less than y. Then, since the last glass drunk may have been either water or wine,

$$u_{x,y} = u_{x-1,y} + u_{x,y-1};$$

$$\therefore \quad u_{x-1,y} = u_{x-2,y} + u_{x-1,y-1},$$
$$u_{x-2,y} = u_{x-3,y} + u_{x-2,y-1},$$
$$\&c. = \&c.;$$
$$u_{y+1,y} = u_{y,y} + u_{y+1,y-1},$$
$$u_{y,y} = 0 + u_{y,y-1}.$$

Therefore, adding and cancelling,

$$u_{x,y} = \sum_{x=y}^{x=x} u_{x,y-1};$$

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$$\therefore \quad u_{x,1} = \sum_{x=1}^{x=x} u_{x,0} = \sum_{x=1}^{x=x} (1) = x, \quad u_{x,2} = \sum_{x=2}^{x=x} x = \frac{(x+1)\,x}{2} - 1,$$

$$u_{x,3} = \sum_{x=3}^{x=x} \left\{ \frac{(x+1)\,x}{2} - 1 \right\} = \frac{(x+2)\,(x+1)\,x}{3!} - (x+2),$$

$$u_{x,4} = \frac{(x+3)\,(x+2)\,(x+1)\,x}{4!} - \frac{(x+3)\,(x+2)}{2!},$$

$$u_{x,5} = \frac{(x+4)\,\ldots\,x}{5!} - \frac{(x+4)\,\ldots\,(x+2)}{3!},$$

and $$u_{x,y} = \frac{(x+y-1)\,\ldots\,x}{y!} - \frac{(x+y-1)\,\ldots\,(x+2)}{(y-2)!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (a);$$

where, in every instance, the constant to be subtracted for the lower limit of x is 0 after $u_{x,3}$. For example, that to be subtracted from $u_{x,y}$ is obtained by putting $x = y-1$ in (a), and is

$$\frac{(2y-2)\,\ldots\,(y-1)}{y!} - \frac{(2y-2)\,\ldots\,(y+1)}{(y-2)!} = 0;$$

therefore $$u_{x,y} = (a) = \frac{(x+y-1)!}{(x+1)!\,y!} \left\{ x\,(x+1) - y\,(y-1) \right\}$$

$$= \frac{(x+y-1)!}{(x+1)!\,y!} (x-y+1)\,(x+y) = \frac{(x-y+1)\,(x+y)!}{(x+1)!\,y!}.$$

Therefore, for the number of ways of drinking n glasses of water and n glasses of wine subject to the condition, put $x = y = n$, and $u_{n,n} = \dfrac{(2n)!}{(n+1)!\,n!}$. But the number of ways of drinking them in random order is $\dfrac{(2n)!}{n!\,n!}$. Therefore the chance that the condition is observed is

$$\frac{2n!}{(n+1)!\,n!} \div \frac{(2n)!}{n!\,n!} = \frac{1}{n+1}.$$

Or the odds against the event are n to 1.

(5804.) The first condition in this Question is the same as that in Quest. 5669, for it is immaterial (mathematically speaking) whether we say that a man never loses more games than he wins, or that he never drinks more wine than water. To express this graphically, let us draw a horizontal line and a vertical line to the left, and divide the included space by vertical and horizontal lines into small spaces, and let us number the columns of spaces 1, 2, 3, ..., $n+r+2$, and the rows 0, 1, 2, 3, ..., $r+1$; and in the space at the intersection of the xth column and the yth row let us enter $u_{x,y}$. Then, if $u_{x,y}$ has the same value as in Quest. 5669, the table (so far as filled up) shows the number of ways of winning x and losing y games subject to the first condition. (See Table I., p. 68.)

But by the second condition the player is never to be more than n games to the good; therefore he cannot win $n+1$ games without having lost 1, or $n+2$ games without having lost 2, and so on; so that he cannot win $n+r$ games without having lost r. Therefore, in the above table, the terms to the right of the zigzag line are non-existent. But this fact will modify the terms from the $n+1$th column onwards which are to the

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Table I.

	1	2	3	...	$n+1$	$n+2$	$n+3$	.....	$n+r$	$n+r+1$	$n+r+2$
0	1	1	1	...	1	1	1	.....	1	1	1
1	1	2	3	...	$u_{n+1,1}$	$u_{n+2,1}$	$u_{n+3,1}$	.....	$u_{n+r,1}$	$u_{n+r+1,1}$	$u_{n+r+2,1}$
2		2	5	...	$u_{n+1,2}$	$u_{n+2,2}$	$u_{n+3,2}$	.....	$u_{n+r,2}$	$u_{n+r+1,2}$	$u_{n+r+2,2}$
3			5	...	$u_{n+1,3}$	$u_{n+2,3}$	$u_{n+3,3}$	.....	$u_{n+r,3}$	$u_{n+r+1,3}$	$u_{n+r+2,3}$
⋮					.....	.....	.....	.....	.....	.....	.....
⋮					.....	.....	.....	.....	.....	.....	.....
⋮					.....	.....	.....	.....	.....	.....	.....
r					$u_{n+1,r}$	$u_{n+2,r}$	$u_{n+3,r}$	.....	$u_{n+r,r}$	$u_{n+r+1,r}$	$u_{n+r+2,r}$
$r+1$					$u_{n+1,r+1}$	$u_{n+2,r+1}$	$u_{n+3,r+1}$	.....	$u_{n+r,r+1}$	$u_{n+r+1,r+1}$	$u_{n+r+2,r+1}$

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Table II.

		$n+1$	$n+2$	$n+3$	$n+4$	.....	$n+r$	$n+r+1$	$n+r+2$
0	.....	1	1	1	1	.....	1	1	1
1	.....	1	$u_{n+2,1}$	$u_{n+3,1}$	$u_{n+4,1}$	.....	$u_{n+r,1}$	$u_{n+r+1,1}$	$u_{n+r+2,1}$
2	.....	1	$u_{n+3,1}$	$u_{n+3,2}$	$u_{n+4,2}$	.....	$u_{n+r,2}$	$u_{n+r+1,2}$	$u_{n+r+2,2}$
3	.....	1	$u_{n+4,1}$	$u_{n+4,2}$	$u_{n+4,3}$	.....	$u_{n+r,3}$	$u_{n+r+1,3}$	$u_{n+r+2,3}$
4	.....	1	$u_{n+5,1}$	$u_{n+5,2}$	$u_{n+5,3}$	.....	$u_{n+r,4}$	$u_{n+r+1,4}$	$u_{n+r+2,4}$
⋮	.....	.....	.....	.....	.....	.....	.....	.....	.....
⋮	.....	.....	.....	.....	.....	.....	.....	.....	.....
r	.....	1	$u_{n+r+1,1}$	$u_{n+r+1,2}$	$u_{n+r+1,3}$	.....	$u_{n+r+1,r-1}$	$u_{n+r+1,r}$	$u_{n+r+2,r}$
$r+1$	.....	1	$u_{n+r+2,1}$	$u_{n+r+2,2}$	$u_{n+r+2,3}$	.....	$u_{n+r+2,r-1}$	$u_{n+r+2,r}$	$u_{n+r+2,r+1}$

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left of that line. Let us therefore make another table, setting down at first all the terms to the right of this line, leaving the spaces to the left blank, and then let us gradually enter on each space to the left the amount by which the number on that particular space must be diminished owing to the non-existence of the terms on the right. (See Table II.)

As the 1 in the top space of the $n+1$th column is non-existent, all the numbers below it in that column are diminished by 1, and 1 is entered for all spaces in that column. Again, the number on space $n+2$, 2 is the sum of that to the left and that over it, and that to the left is diminished by 1 and that over it is now non-existent; therefore the number on $n+2$, 2 is diminished by $1+u_{n+2,1}$ or $u_{n+3,1}$. Similarly the numbers to be entered on $n+2$, 3 and $n+2$, 4 are $u_{n+4,1}$ and $u_{n+5,1}$. Again, the numbers to be entered on $n+3$, 3 and $n+3$, 4, &c., are $u_{n+4,2}$, $u_{n+5,2}$, &c., and on $n+4$, 4, &c., are $u_{n+5,3}$, &c. Now it will be observed that the terms of rows 1, 2, 3, 4 within the zigzag line are identical with the terms (excluding the last) of columns $n+2$, $n+3$, $n+4$, $n+5$ outside that line. Suppose this law to hold for the rth row, so that its terms are identical with those (excluding the last) of the $n+r+1$th column outside the line. Then, if 1 is the first term of the $r+1$th row, all the other terms in that row are derived from the rth row by the formula $u_{x,y} = u_{x-1,y} + u_{x,y-1}$, and are identical with the terms (excluding the last) of the $n+r+2$th column outside the line. If, therefore, this law of substitution holds for the rth row, it holds for the $r+1$th row. But it is seen to hold when $r = 1, 2, 3, 4$. Therefore it holds for all integral values of r. Therefore, when *both* the limitations of the Question are observed, the entry on any space in Table I. is to be diminished by the entry on the corresponding space in Table II.

Now the entry on space $n+r+1$, $r+1$ is $u_{n+r+1,r+1}$ in Table I., and $u_{n+r+2,r}$ in Table II. Therefore, putting $r = n$, the number of ways of winning $2n+1$ and losing $n+1$ games subject to both conditions is

$$u_{2n+1,n+1} - u_{2n+2,n}$$

$$= \frac{(n+1)(3n+2)!}{(2n+2)!\,(n+1)!} - \frac{(n+3)(3n+2)!}{(2n+3)!\,n!}$$

$$= \frac{(3n+2)!}{(2n+1)!\,(n+1)!}\left\{\frac{1}{2} - \frac{n+3}{2(2n+3)}\right\} = \frac{(3n+2)!}{(2n+1)!\,(n+1)!} \cdot \frac{n}{4n+6} \quad \ldots (a).$$

But the number of ways of winning $2n+1$ games and losing $n+1$ games in random order is

$$\frac{(3n+2)!}{(2n+1)!\,(n+1)!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Therefore the chance that the conditions are observed is

$$(a) \div (b), \quad \text{or} \quad \frac{n}{4n+6}.$$

5882. (Professor Cochez.)—Calculer le rayon d'un cercle inscrit dans un triangle en fonctions des distances du centre aux trois sommets.

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Solution by F. E. Cave.

Let r be the radius of inscribed circle, α, β, γ the distances of its centre from the angular points. Let l, m, n denote $s-a, s-b, s-c$. Then

$$r^2 = \frac{lmn}{l+m+n},$$

and $$l = \sqrt{(\alpha^2-r^2)}, \quad r^2 = \frac{\sqrt{(\alpha^2-r^2)(\beta^2-r^2)(\gamma^2-r^2)}}{\sqrt{(\alpha^2-r^2)}+\sqrt{(\beta^2-r^2)}+\sqrt{(\gamma^2-r^2)}}.$$

Hence we easily get $4r^6\alpha^2\beta^2\gamma^2 = (r^2\Sigma\alpha^2\beta^2-\alpha^2\beta^2\gamma^2)^2$;

therefore $2r^3\alpha\beta\gamma = \pm(r^2\Sigma\alpha^2\beta^2-\alpha^2\beta^2\gamma^2)$.

Choosing the sign so as to make r positive in the case of an equilateral triangle ($r = \frac{1}{2}\alpha$),

$$2r^3\alpha\beta\gamma + r^2\Sigma\alpha^2\beta^2 - \alpha^2\beta^2\gamma^2 = 0 \quad \ldots\ldots\ldots\ldots (1).$$

This has one, and only one, real positive root.

Thus the required value is the real positive root of (1).

14036. (R. W. D. Christie.) — Explain the following instance of curious figuring:— $151 \cdot 107 - 1 = 109$ (M).

M is always the series 1, 1, 2, 3, 5, 8, 13, ..., placed in reversed order, *i.e.*, $= 109\,(13853211)$.

Similarly with other values not accidental.

Solution by H. W. Curjel, M.A.

Let $$S_n = u_1 + u_2 r + u_3 r^2 + \ldots + u_n r^{n-1},$$

where the u's are the terms of the series 1, 1, 2, 3, 5, 8, ..., so that

$$u_{n+1} = u_n + u_{n-1}.$$

Then $S_n(r^2+r-1) = r^n(u_n + u_{n-1} + ru_n) - 1 = r^n(u_{n+1} + ru_n) - 1$.

The example is a particular case: $n = 7$, $r = 10$.

[The Proposer solves the Question as follows:—This Question is a particular case ef the more general theorem (see Quest. 13980)

$$a \cdot 10^n - 1 = \mathrm{P}(\mathrm{M}).$$

Let $$S_n = m^{n-1} + \frac{n-2}{1!}m^{n-3} + \frac{n-3 \cdot n-4}{2!}m^{n-5} + \&c.,$$

where $$u_{n+1} = m \cdot u_n + u_{n-1}.$$

Then $(r \cdot u_n + u_{n+1})r^n - 1 = (r^2 + mr - 1)(u_n r^{n-1} + u_{n-1} r^{n-2} + \ldots + 1)$

$$= (r^2 + mr - 1)(S_n).$$

In the example given, $m = 1$, $n = 7$, $r = 10$.

Another example is $m = 3$, $n = 4$, $r = 10$, $u_n = 1, 3, 10, 33$, &c., and we get $$(10 \times 33 + 109)\,10^4 - 1 = (10^2 + 3 \times 10 - 1)(34031),$$

where $(\mathrm{M}) = S_n = 34031$ possesses the peculiar property mentioned.]

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13918. (R. KNOWLES, B.A.)—A fixed right line (not the axis) is drawn through the vertex of a parabola, and tangents are drawn from points in this line to the curve; prove that the locus of the centres of circles through each point and its points of contact is a parabola.

Solution by Professor A. RADHAKRISHNAN.

Let $y^2 = 4ax$ represent the parabola, and $y = mx$ the given straight line.

The polar of (x', mx') is $mx'y - 2a(x + x') = 0$.

Any circle through the points of contact will be given by

$$k(y^2 - 4ax) + \{mx'y - 2a(x + x')\}(mx'y + 2ax + c) = 0,$$

where $$k + m^2x'^2 = -4a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For the circle to pass through (x', mx') also

$$k(m^2x'^2 - 4ax') + (m^2x'^2 - 4ax')(m^2x'^2 + 2ax' + c) = 0,$$

or $$k + m^2x'^2 + 2ax' + c = 0,$$

or $$-4a^2 + 2ax' + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substituting for k and c from (1) and (2), the equation to the circle becomes

$$-4a^2(x^2 + y^2) + 4ax(m^2x'^2 + 4a^2) + (mx'y - 2ax - 2ax')(4a^2 - 2ax') - 2ax'(mx'y + 2ax) = 0,$$

or $$x^2 + y^2 - x/a(m^2x_1^2 + 2a^2) - (1 - x'/a)mx'y + \ldots = 0;$$

or, if (x, y) be the centre of that circle,

$$2x = (m^2x_1^2 + 2a^2)/a, \quad 2y(1 - x'/a)mx' = mx' - 1/m(2x - 2a);$$

therefore $$\{2y + 1/m(2x - 2a)\}^2 = 2ax - 2a^2,$$

which is the equation to a parabola.

14191. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Find the roots of the quadratic equation

$$8x^2 - 2(n^4 + 2n^3 + 3n^2 + 2n)x + n^6 + 3n^5 + 3n^4 + n^3 = 0,$$

and exhibit each as a *series*.

Solution by JAMES MCCUBBIN, B.A.; R. TUCKER, M.A; *and others.*

$$8x^2 - 2(n^4 + 2n^3 + 3n^2 + 2n)x = -(n^6 + 3n^5 + 3n^4 + n^3).$$

Then $$x = \tfrac{1}{16}\Big[2(n^4 + 2n^3 + 3n^2 + 2n) \pm \sqrt{\{2(n^4 + 2n^3 + 3n^2 + 2n)\}^2 - 32(n^6 + 3n^5 + 3n^4 + n^3)}\Big]$$

$$= \tfrac{1}{16}\{2(n^4 + 2n^3 + 3n^2 + 2n) \pm 2(n^4 + 2n^3 - n^2 - 2n)\}$$

$$= \big[\tfrac{1}{2}\{n(n+1)\}\big]^2 \quad \text{or} \quad \tfrac{1}{2}\{n(n+1)\}$$

$$= 1^3 + 2^3 + 3^3 + \ldots + n^3 \quad \text{or} \quad 1 + 2 + 3 + \ldots + n$$

$$= \Sigma(n^3) \quad \text{or} \quad \Sigma(n).$$

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14105. (Professor CROFTON, F.R.S.)—If $u = x^2 + ax + b$, and we put for shortness the symbol $\mathrm{D}u\mathrm{D} = \omega$, then

$$\mathrm{D}^n u^n \mathrm{D}^n \equiv \omega\,(\omega - 1\,.\,2)\,(\omega - 2\,.\,3)\,\ldots\,\{\omega - (n-1)\,n\}.$$

Solution by Rev. J. CULLEN.

It is clearly only necessary to show that

$$\mathrm{D}^n u^n \mathrm{D}^n v = \mathrm{D}^{n-1} u^{n-1} \mathrm{D}^{n-1} \{\mathrm{D}u\mathrm{D} - (n-1)\,n\}\,v,$$

where v is any function of x. Now

$$\mathrm{D}^n u^n \mathrm{D}^n v = \mathrm{D}^{n-1}\,.\,\mathrm{D}u^n \mathrm{D}^n v = \mathrm{D}^{n-1}\{nu^{n-1}\mathrm{D}u\,.\,\mathrm{D}^n v + u^n \mathrm{D}^{n+1} v\}\,;$$

also

$$\mathrm{D}^{n-1} u^{n-1} \mathrm{D}^{n-1} \{\mathrm{D}u\mathrm{D} - (n-1)\,n\}\,v = \mathrm{D}^{n-1}\{u^{n-1}\mathrm{D}^n u \mathrm{D}v - (n-1)\,n\mathrm{D}^{n-1}v\}\,;$$

$$= \mathrm{D}^{n-1}\{u^n \mathrm{D}^{n+1} v + nu^{n-1}\mathrm{D}u\,.\,\mathrm{D}^n v + \tfrac{1}{2}(n-1)nu^{n-1}\mathrm{D}^2 u \mathrm{D}^{n-1} v - (n-1)n\mathrm{D}^{n-1}v\}$$

but $\mathrm{D}^2 u = 2\,;$

therefore, &c.

13993. (C. E. HILLYER, M.A.)—AD, BE, CF are the altitudes, H the orthocentre, and I the incentre of a triangle ABC. Prove that the three chords of intersection of the circles AID, BIE, CIF with the incircle meet in the mid-point of IH.

Solution by Professor A. DROZ-FARNY *and* LIONEL E. REAY, B.A.

On sait que

$$(\mathrm{OH})^2 = \mathrm{R}^2 - 8\mathrm{R}^2 \cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}$$
$$= 9\mathrm{R}^2 + 2r^2 + 8\mathrm{R}r - 2p^2,$$

donc

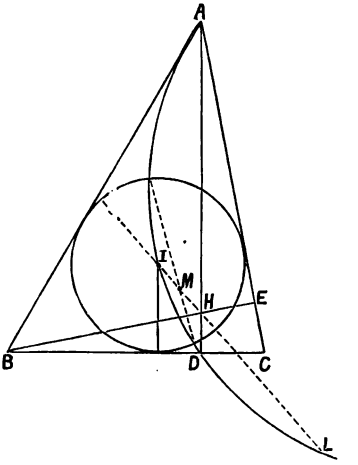
$$4\mathrm{R}^2 \cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C} = p^2 - 4\mathrm{R}^2 - r^2 - 4\mathrm{R}r \ldots \text{(i.)},$$

et

$$(\mathrm{IH})^2 = 4\mathrm{R}^2 + 4\mathrm{R}r + 3r^2 - p^2 \ldots \text{(ii)}.$$

Soit M le point milieu de IH, et représentons par L le second point de coupe de IH avec la circonférence AID. La puissance P de M par rapport au cercle inscrit sera

$$\mathrm{P} = (r + \mathrm{IM})(r - \mathrm{IM})$$
$$= r^2 - \tfrac{1}{4}(\mathrm{IH})^2.$$



La puissance P′ de M par rapport au cercle AID sera

$$\mathrm{P}' = \tfrac{1}{2}\mathrm{IH}\,(\tfrac{1}{2}\mathrm{IH} + \mathrm{HL})$$
$$= \tfrac{1}{4}(\mathrm{IH})^2 + \tfrac{1}{2}(\mathrm{IH}\,.\,\mathrm{HL})$$
$$= \tfrac{1}{4}(\mathrm{IH})^2 + \tfrac{1}{2}(\mathrm{AH}\,.\,\mathrm{HD}) = \tfrac{1}{4}(\mathrm{IH})^2 + 2\mathrm{R}^2 \cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}.$$

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Or, en vertu des formules (i.) et (ii.),

$$P = P' = \tfrac{1}{4}(p^2 + r^2 - 4R^2 - 4Rr).$$

La corde d'intersection du cercle inscrit et du cercle AID passe donc par M.

[The PROPOSER solves the Question as follows:—

It is known that $IH^2 + AH.HD = 2r^2$. But $AH.HD = IH.HL$. Thus $IH.IL = IH^2 + IH.HL = 2r^2$, and therefore $IM.IL = r^2$; therefore the chord of intersection of the incircle with the circle AID passes through M.

Remark.—Two geometrical proofs of the theorem $IH^2 + AH.HD = 2r^2$ are given by Mr. LEUDESDORF in an article entitled "Proofs of FEUERBACH's Theorem" (*Messenger of Mathematics*, Vol. XIII., p. 117). The following is another proof.

1. $I_1H^2 - IH^2 = 2r_1^2 - 2r^2$.

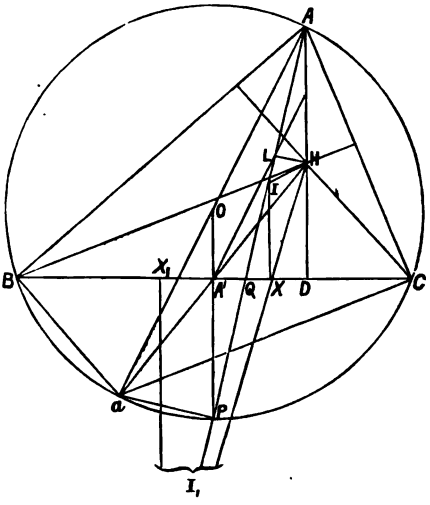
If AO meet the circumcircle in a, aBA is a right angle; therefore aB is parallel to CH, and similarly aC is parallel to BH; therefore $HBaC$ is a parallelogram; therefore Ha and BC bisect each other in A'. P is the mid-point of II_1, and aPA is a right angle; therefore $aI = aI_1$,

$IH^2 + Ia^2 = 2IA'^2 + 2HA'^2$,

and $I_1H^2 + I_1a = 2I_1A'^2 + 2HA'^2$;

therefore $I_1H^2 + IH^2$
$= 2I_1A'^2 - 2IA'^2$
$= 2(I_1X_1^2 + A'X_1^2) - 2(IX^2 + A'X^2)$
$= 2r_1^2 - 2r^2$,

since $A'X_1 = A'X$.



2. $IH^2 + AH.HD = 2r^2$.

Draw HL perpendicular to AI, and let AI meet BC in Q; then, since $AI/AI_1 = QI/QI_1 = r/r_1$ and P is the mid-point of II_1, we have

$PA/PI = (r_1 + r)/(r_1 - r)$, $PQ/PI = (r_1 - r)/(r_1 + r)$, $PQ.PA = PI^2$,

also $4PL.PI = I_1L^2 - IL^2 = I_1H^2 - IH^2 = 2r_1^2 - 2r^2$

from (1); therefore $2PL.PI = r_1^2 - r^2$.

Thus $PI/(r_1 - r) = (r_1 + r)/2PL = PA/(r_1 + r)$,

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and $\quad PI/(r_1+r) = (r_1-r)/2PL = PQ/(r_1-r)$;

therefore $\quad 2PA.PL = (r_1+r)^2$ and $2PQ.PL = (r_1-r)^2$.

Now $AH.HD+IH^2$

$= AH.AD-(AH^2-IH^2)$

$= AL.AQ-(AL^2-IL^2)$

(since H, D, Q, L are concyclic)

$= AL.LQ+IL^2 = (PA-PL)(PL-PQ)+(PI-PL)^2$

$= PA.PL+PL.PQ-2PI.PL$

$= \frac{1}{2}(r_1+r)^2+\frac{1}{2}(r_1-r)^2-(r_1^2-r^2) = 2r^2$.]

14119. (Rev. W. Allen Whitworth, M.A.)—If out of m cards numbered 1, 2, 3, ..., m we draw n cards at random, the expectation of the lowest drawn is $(m+1)/(n+1)$ and the expectation of the highest is $n(m+1)/(n+1)$.

[Corollary.—If n numbers be named at random, the expectation of the highest is n times the expectation of the lowest.]

Solution by H. W. Curjel, M.A.

The chance that x is the highest number is

$$({}^xC_n - {}^{x-1}C_n)/{}^mC_n;$$

therefore the expectation of the highest number is

$$\{m\,{}^mC_n - {}^{m-1}C_n - {}^{m-2}C_n - \ldots - {}^nC_n\}/{}^mC_n$$

$$= \left\{(m+1)\,{}^mC_n - 1 - \frac{n+1}{1} - \frac{(n+1)(n+2)}{2!} - \ldots - \frac{(n+1)(n+2)\ldots m}{(m-n)!}\right\} \Big/ {}^mC_n$$

$$= \{(m+1)\,{}^mC_n - {}^{m+1}C_{n+1}\}/{}^mC_n$$

$$= m+1-(m+1)/(n+1) = \{n(m+1)\}/(n+1).$$

The chance that x is the highest number clearly equals the chance that $m+1-x$ is the lowest; therefore expectation of the lowest is

$$m+1-\{n(m+1)\}/(n+1) = (m+1)/(n+1).$$

Hence, if the numbers must be all different, the expectation of the highest number is n times the expectation of the lowest, whatever m may be.

14004. (Robert W. D. Christie.)—Let N, an integer, consist of p units when p is a prime >3. Prove (1) N is of form $pq+1$; (2) every factor of N is of like form; (3) the sum of the digits of one factor $= pr$.

Ex. gr. $\quad 11111111111 = 21649 \times 513239 = (px+1)(py+1)$

and $\quad 2+1+6+4+9 = 11 \times 2.$

Show incidentally that the factors of q in (1) can be written down at sight.

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Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Here $N = \frac{10^p-1}{10-1}$, and properties (1) and (2), that $N = pq+1$ and that every factor of $N = pq'+1$, have been proved by EULER (see *Comment. Arithmet.*, Petrop., 1849, Vol. I., Mem. vii., Art. 38.) Property (3) is *not true* in general of the *prime* factors; *e.g.*,

$$(10^{13}-1) + (10-1) = 53 \cdot 79 \cdot 265{,}371{,}653;$$

and none of these three *prime* factors has the sum of digits equal to a multiple of 13. Similarly, it is not true of any *prime* factor of

$$(10^{29}-1) + (10-1).$$

Lastly, $$q = \frac{1}{p}(N-1) = \frac{1}{p}\left(\frac{10^p-1}{10-1}-1\right) = \frac{10^p-10}{(10-1)p}$$

$$= 10\frac{(10^{\varpi}-1)(10^{\varpi}+1)}{(10-1)p}, \text{ where } \varpi = \tfrac{1}{2}(p-1),$$

whereby several factors of q are evident [*one* of $(10^{\varpi} \mp 1)$ contains p].

[The PROPOSER remarks as follows:—"The SOLVER says: 'Property (3) is *not true* in general of the *prime* factors, *e.g.*,

$$(10^{13}-1)+(10-1) = 53 \times 79 \times 265{,}371{,}653.'$$

This property was not predicated of the prime factors; but it is *in general* true nevertheless; *e.g.*,

$$111 = 37 \times 3 \text{ where } 1+1+1 = 3r,$$

$$11111 = 41 \times 271 \text{ where } 4+1 = 2+7+1 = 5r,$$

$$1111111 = 239 \times 4649 \text{ where } 2+3+9 = 7r,$$

$$11111111111 = 21649 + 513239 \text{ where } 2+1+6+4+9 = 11r;$$

and, finally, $$1111111111111 = 79 \times 14064697609$$

where $$1+4+0+6+4+6+9+7+6+0+9 = 13r,$$

In exceptional cases where N = prime, then digits of $N = pr$.

$(10^{\varpi}-1)+(10-1)$ provides *all* the factors of q, since $N_1 = (10^{\varpi}-1)+(10-1)$, where ϖ is some prime of lower order than p."]

14170. (Professor SANJÁNA, M.A.)—Let Q be the orthocentre of the pedal triangle of ABC; let S, K, T be the circum-, cosine-, and TAYLOR-centres of ABC. Prove that S, K, T, Q are collinear; that T bisects SQ, and that K divides it in the ratio $1 : 1+2\cos A \cos B \cos C$.

Solution by R. TUCKER, M.A.

In the Appendix to Vol. XV. of the *Proceedings of the London Mathematical Society* (p. 280, Christmas, 1884), I have shown that the α (trilinear) coordinates of Q (my J) and T (my ω) are respectively

$$-R\cos 2A \cos(B-C) \quad \text{and} \quad R\{\cos A - \cos 2A \cos(B-C)\}/2,$$

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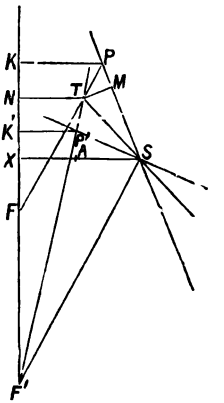
and that these points lie on the circum-Brocardal axis. (See also MILNE's *Companion*, p. 145, and CASEY's *Sequel*, 5th edition, p. 241.) It is at once seen that T is mid-point of SQ. For the last property, see MILNE (*l.c.*, p. 145, Quest. 6, by Mr. R. F. DAVIS).

5505. (CHRISTINE LADD.)—If, from two points on the directrix, which subtend a constant angle at the focus, tangents to a conic be drawn, they will intersect on another conic having the same focus and directrix.



Solution by Rev. J. CULLEN.

Let FP and F'P' be two tangents intersecting in T. Draw TM and TN at right angles to SP and FK. Then FS and F'S are at right angles to SP and SP'. Hence $\angle PSP' = \angle FSF' =$ constant. Also TS bisects $\angle PSP'$; therefore $\angle TSM =$ constant; so that SM/ST is given. Also

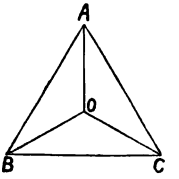
$$SM : TN :: SA : SX;$$

therefore $ST = k\,TN$,

which proves the proposition.

5276. (Professor EVANS, M.A.)—ABC is a plane triangle, and OA, OB, OC are lines making equal angles with one another. Find the least integral values of BC, CA, AB that will make OA, OB, OC integral.

Solution by the PROPOSER.



Let ABC be the triangle; a, b, c its sides,

$$CO = x, \quad BO = y, \quad AO = z.$$

Then $a^2 = x^2 + xy + y^2$, $b^2 = x^2 + xz + z^2$,

$c^2 = y^2 + yz + z^2$.

Solving these equations, we get

$$x = \frac{a^2+b^2-c^2 \pm \frac{1}{3}\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}}{\left[2\,(a^2+b^2+c^2) \pm 2\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}},$$

$$y = \frac{a^2+c^2-b^2 \pm \frac{1}{3}\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}}{\left[2\,(a^2+b^2+c^2) \pm 2\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}},$$

$$z = \frac{b^2+c^2-a^2 \pm \frac{1}{3}\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}}{\left[2\,(a^2+b^2+c^2) \pm 2\left\{12a^2b^2 - 3\,(a^2+b^2-c^2)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}},$$

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Let $c = (1+n)\,b$, $a = (1-n)\,b$; then, since $a^2 + b^2 > c^2$,

$$2 - 2n + n^2 > 1 + 2n + n^2 \quad \text{or} \quad n < \tfrac{1}{4}.$$

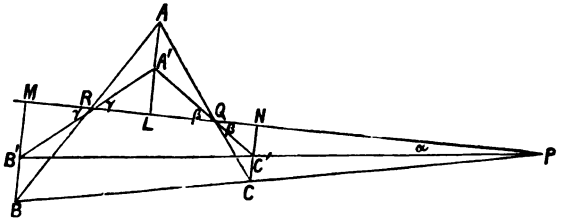
Take $n = \frac{5}{65}$, and, using the upper sign, we get $x = \frac{195}{455}b$, $y = \frac{264}{455}b$, $z = \frac{325}{455}b$, $a = \frac{399}{455}b$, $b = \frac{392}{455}b$, $c = \frac{511}{455}b$. When $b = 455$, we have $a = 399$, $b = 455$, $c = 511$, $x = 195$, $y = 264$, $z = 325$.

14195. (Professor Neuberg.)—Soit A′B′C′ la projection du triangle ABC sur un plan faisant l'angle ϕ avec le plan ABC. Démontrer que les angles des deux triangles vérifient la relation

$$\cos\phi + \sec\phi = \cot A \cot B' + \cot B \cot C' + \cot C \cot A' + \cot A' \cot B + \cot B' \cot C + \cot C \cot A.$$

Solution by R. F. Davis, M.A.; *and* F. L. Ward, B.A.

Let ABC be a triangle; PQR a transversal intersecting BC (produced), CA, AB in P, Q, R respectively. Draw AL, BM, CN perpendiculars on PQR; and let $A'L : AL = B'M : BM = C'N : CN = \cos\phi : 1$, so that A′B′C′ may be considered the orthogonal projection of ABC.



Obviously the sides of A′B′C′ pass through P, Q, R. Denote the angles C′PN, A′QL, B′RM by α, β, γ respectively.

Then $$AL^2 - RL.QL = QR.AL \cot A.$$
$$AL'^2 - RL.QL = QR.A'L \cot A'.$$

Therefore $$A'L^2(\sec^2\phi - 1) = A'L.QR\,(\sec\phi \cot A - \cot A'),$$

or $$\cot\beta + \cot\gamma = \tan^2\phi/(\sec\phi \cot A - \cot A').$$

Similarly, $$-\cot\gamma + \cot\alpha = \ldots\,; \quad -(\cot\alpha + \cot\beta) = \ldots$$

Hence $$(\sec\phi \cot\alpha - \cot A')^{-1} + \ldots + \ldots = 0\,;$$

this, when simplified, gives required result, remembering

$$\Sigma \cot B \cot C = \Sigma \cot B' \cot C' = 1.$$

If $A' = B' = C' = 60°$, we have $\cos\phi + \sec\phi = 2\cot\omega/\sqrt{3}$,

where ω is the Brocard angle of ABC. This equation is always possible since $\cot\omega > \sqrt{3}$. It is well known that a triangle will be projected into an equilateral triangle when the maximum ellipse inscribed in the first triangle (touching the sides at its middle points) is projected into a circle.

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Let a, β be the semi-axes of this ellipse; then it can be easily shown that

$$a^2+\beta^2 = (a^2+b^2+c^2)/18 = 2\Delta \cot \omega/9, \quad a\beta = \Delta/3\sqrt{3}.$$

Hence $$\cos\phi + \sec\phi = \beta/a + a/\beta = 2\cot\omega/\sqrt{3},$$

as before.

This matter has already been discussed (prior to the invasion of the BROCARD geometry) in the *Reprint*, Vol. XXIX., p. 83; Vol. XXXI., pp. 18–21; Vol. XXXVI., p. 64.

14181. (H. MACCOLL, B.A.)—Let A and B be two events which are not mutually independent. Show that, if the dependence of A upon B is equal to the dependence of B upon A, then the chance of A happening is equal to the chance of B happening. *Definitions.*—The fractional symbol a/β (when a and β are statements) denotes the chance that a is true *on the assumption that β is true*; the symbol a/ϵ denotes simply the chance that a is true, *nothing being taken for granted but the data of the problem.* The symbol $\delta\, a/\beta$ denotes $a/\beta - a/\epsilon$, and is called the *dependence of a upon β*, or of the event A, asserted by a, upon the event B, asserted by β.

Solution by the PROPOSER.

Let the letter z as exponent convert any expression ϕ affected by it into a *statement* which asserts that ϕ is *zero*. Thus, $(x-y)^z$ asserts that $x = y$.

The colon symbol (:) is short for the word *implies*. Let $a = \frac{A}{\epsilon}$, and let $b = \frac{B}{\epsilon}$. We have to prove

$$\left(\delta\frac{A}{B} - \delta\frac{B}{A}\right)^z : (a-b)^z.$$

Assuming the easily proved formulæ $\frac{A}{B} = \frac{a}{b}\cdot\frac{B}{A}$ and $\delta\frac{A}{B} = \frac{a}{b}\,\delta\frac{B}{A}$,

we get $$\left(\delta\frac{A}{B} - \delta\frac{B}{A}\right)^z : \left(\frac{a}{b}\,\delta\frac{B}{A} - \delta\frac{B}{A}\right)^z : \left(a\delta\frac{B}{A} - b\delta\frac{B}{A}\right)^z$$

$$: \left\{(a-b)\,\delta\frac{B}{A}\right\}^z : (a-b)^z.$$

for $\delta\frac{B}{A}$ cannot be zero, since, by hypothesis, A and B are not independent. Hence, finally, $$\left(\delta\frac{A}{B} - \delta\frac{B}{A}\right)^z : (a-b)^z.$$

Note.—The symbol $\delta\frac{a}{\beta}$ (which expresses the amount of dependence of a upon β) indicates the increase or (when negative) diminution which the chance $\frac{a}{\epsilon}$ undergoes when β is added to our data ϵ. When we find this to be zero, we say that a is independent of β, which implies that β is

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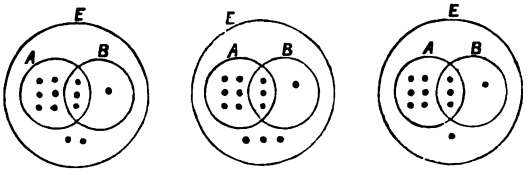
independent of α. The meanings of the symbols $\frac{A}{B}$ and $\delta\frac{A}{B}$, when A and B are statements, will appear from the accompanying three figures:



In Fig. 1, $\delta\frac{A}{B} = \frac{A}{B} - \frac{A}{\epsilon} = \frac{3}{4} - \frac{9}{12} = 0.$

In Fig. 2, $\delta\frac{A}{B} = \frac{A}{B} - \frac{A}{\epsilon} = \frac{3}{4} - \frac{9}{13} = \frac{3}{52}.$

In Fig. 3, $\delta\frac{A}{B} = \frac{A}{B} - \frac{A}{\epsilon} = \frac{3}{4} - \frac{9}{11} = -\frac{3}{44}.$

Out of the total number of points marked in the circle E (taking any of the three figures) a point P is taken at random. The symbol $\frac{A}{\epsilon}$ denotes the chance that P will be one of the nine points in the circle A; $\frac{B}{\epsilon}$ denotes the chance that P will be one of the four points in the circle B; and $\frac{A}{B}$ denotes the chance that P will be one of the points in the circle A when it is taken at random—not out of the total number in E, but out of the four in B.

[Mr. Curjel solves the Question thus:

$$\frac{\alpha}{\beta} - \frac{\alpha}{\epsilon} = \delta\,\alpha/\beta = \delta\,\beta/\alpha = \frac{\beta}{\alpha} - \frac{\beta}{\epsilon}; \quad \text{therefore} \quad \frac{\alpha}{\beta} + \frac{\beta}{\epsilon} = \frac{\beta}{\alpha} + \frac{\alpha}{\epsilon}.$$

But $\left(\frac{\beta}{\epsilon}\right)\left(\frac{\alpha}{\beta}\right)$ = chance that both A and B will happen = $\left(\frac{\alpha}{\epsilon}\right)\left(\frac{\beta}{\alpha}\right)$;

therefore $\frac{\alpha}{\beta} = \frac{\beta}{\alpha}$ and $\frac{\beta}{\epsilon} = \frac{\alpha}{\epsilon}$, *i.e.*, chance of A happening equals chance of B happening, for $\frac{\alpha}{\beta} = \frac{\alpha}{\epsilon}$ and $\frac{\beta}{\epsilon} = \frac{\beta}{\alpha}$ would imply that A and B are mutually independent.]

14243. (G. H. Hardy, B.A.)—If p, m are integers, $p > m$, then, if m is odd,

$$\int_0^\pi x\cos px\sin^m x\,dx = (-)^{\frac{1}{2}[m+(-1)^p]}\left\{\frac{m!\,\pi}{(p^2-1^2)(p^2-3^2)\ldots(p^2-m^2)}\right\};$$

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and, if m is even,

$$\int_0^\pi x \sin px \sin^m x\, dx = (-)^{\frac{1}{2}[m+1+(-1)^p]} \left\{ \frac{m!\,\pi}{p(p^2-2^2)(p^2-4^2)\ldots(p^2-m^2)} \right\}.$$

Solution by J. H. Dibb, B.Sc.

We know that $\int_0^\pi \sin px \sin^m x\, dx$ vanishes when m is odd, and $\int_0^\pi \cos px \sin^m x \cos x\, dx$, when m is even. Let $f(m)$ denote

$$\int_0^\pi x \cos px \sin^m x\, dx.$$

Then, integrating by parts, we get

$$f(m) = -\frac{1}{p}\int_0^\pi \sin px\,(\sin^m x + mx \sin^{m-1} x \cos x)\, dx$$

$$= -\frac{m}{p}\int_0^\pi x \sin^{m-1} x \cos x \sin px\, dx.$$

Again integrating by parts,

$$f(m) = -\frac{m}{p^2}\int_0^\pi \cos px \left\{ \sin^{m-1} x \cos x + (m-1)\, x \sin^{m-2} x - mx \sin^m x \right\} dx$$

$$= -\frac{m(m-1)}{p^2} f(m-2) + \frac{m^2}{p^2} f(m);$$

therefore $$f(m) = -\frac{m(m-1)}{p^2-m^2} f(m-2)$$

$$= (-)^{\frac{1}{2}(m-1)} \frac{m(m-1)\ldots 2}{(p^2-m^2)\left\{p^2-(m-2)^2\right\}\ldots(p^2-3^2)} f(1).$$

But $$f(1) = \int_0^\pi x \cos px \sin x\, dx$$

$$= \tfrac{1}{2}\int_0^\pi x \sin(p+1)\, x\, dx - \tfrac{1}{2}\int_0^\pi x \sin(p-1)\, x\, dx = \frac{(-1)^{p+1}\pi}{p^2-1^2}.$$

Considering separately the cases of p odd and even, we see that

$$(-1)^{\frac{1}{2}(m-1)}(-1)^{p+1} = (-1)^{\frac{1}{2}[m+(-1)^p]};$$

therefore $$f(m) = (-1)^{\frac{1}{2}[m+(-1)^p]} \left\{ \frac{m!\,\pi}{(p^2-1^2)(p^2-3^2)\ldots(p^2-m^2)} \right\}.$$

The second formula is established in a precisely similar manner.

N.B.—*When p is even* the first formula is easily obtained by the well known transformation

$$f(m) = \int_0^\pi (\pi - x) \cos p(\pi - x) \sin^m (\pi - x)\, dx$$

$$= \pi \int_0^\pi \cos px \sin^m x\, dx - f(m).$$

A similar method can be applied to the second formula *when p is odd.*

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14168. (The late Professor WOLSTENHOLME, D.Sc.)—Given a quadric ($u = 0$) and two fixed points O, O′, S, S′ denote the plane sections of u by the polar planes of O, O′; prove that, if any straight line through O meet the quadric in points P, P′, the two cones whose vertices are P, P′ and base the plane section S′ will intersect in a second plane section whose locus is the quadric ($v = 0$) containing S, S′, and such that O is the pole of S′, and O′ the pole of S. So also, if QQ′ be any chord of u passing through O′, the locus of the second plane of intersection of the cones whose vertices are Q, Q′ and base S is the same quadric.

Solution by Rev. J. CULLEN.

Let $u \equiv Ax^2 + By^2 + Cz^2 - 1 = 0$, O (x', y', z'), O′ (x'', y'', z''), P (α, β, γ) and P′ $(\alpha', \beta', \gamma')$. Then

$$S \equiv \Sigma Axx' - 1 = 0, \quad S' \equiv \Sigma Axx'' - 1 = 0.$$

Also u and S′ may be written in the forms

$$u = \Sigma A(x-\alpha)^2 + 2\Sigma A(x-\alpha)\alpha = 0, \quad S' \equiv \Sigma A(x-\alpha)x'' + \Sigma A\alpha x'' - 1 = 0.$$

Therefore the cone having P for the vertex and S′ for the base is

$$\Sigma A(x-\alpha)^2(\Sigma A\alpha x'' - 1) - 2\Sigma A(x-\alpha)x'\Sigma A(x-\alpha)\alpha = 0,$$

or

$$u(\Sigma A\alpha x'' - 1) - 2S'(\Sigma A\alpha x - 1) = 0 \quad \ldots\ldots(1).$$

In like manner the cone having P′ for the vertex and the same base is

$$u(\Sigma A\alpha' x' - 1) - 2S'(\Sigma A\alpha' x - 1) = 0 \quad \ldots\ldots(2);$$

whence we easily find the second plane of intersection to be

$$(\Sigma Ax'x'' - 1)(\Sigma A\alpha x - 1) - S(\Sigma A\alpha x'' - 1) = 0 \quad \ldots\ldots(3).$$

Therefore from (1) and (3), we have $v \equiv uk - 2SS' = 0$, where $k = 0$ is the condition that O (or O′) should lie on S′ (or S).

Hence v clearly contains S and S′, and, forming the polar plane in the ordinary manner, we see that O is the pole of S′, and O′ that of S.

The final result in the Question is obtained by interchanging x', &c., and x'', &c., throughout in the above equations.

[The PROPOSER made the following remarks on this Question:—If $(x_1y_1z_1w_1)$, $(x_2y_2z_2w_2)$ be the two points O, O′, the equation of the quadric locus is

$$\left(x_2\frac{\partial u_1}{\partial x_1} + y_2\frac{\partial u_1}{\partial y_1} + z_2\frac{\partial u_1}{\partial z_1} + w_2\frac{\partial u_1}{\partial w_1}\right)u = \left(x_1\frac{\partial u}{\partial x} + \ldots\right)\left(x_2\frac{\partial u}{\partial x} + \ldots\right),$$

and the invariant relation between this and u is $\Theta^2\Delta' + \Theta'^2\Delta = \Theta\Theta'\Phi$; the discriminant of $ku + v$ being $\Delta k^4 + \Theta k^3 + \Phi k^2 + \Theta' k + \Delta'$. Two other points $(X_1Y_1Z_1W_1)$, $(X_2Y_2Z_2W_2)$ may also be found such that

$$v \equiv u - \frac{\left(X_1\frac{\partial u}{\partial x} + Y_1\frac{\partial u}{\partial y} + Z_1\frac{\partial u}{\partial z} + W_1\frac{\partial u}{\partial w}\right)^2}{4U_1} - \frac{\left(X_2\frac{\partial u}{\partial x} + \ldots\right)^2}{4U_2},$$

where U_1, U_2 denote the values of u at the two points. The quadric v may be generated from u, and these last two points by a mode reciprocal to the former, and the form of the equation shows that it touches each of the tangent axes drawn to u from the two points along the plane section made by the plane of contact with u of the other.]

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13999. (R. KNOWLES, B.A.)—Prove that (1) the $(r+1)$th term in the development of

$$\left\{1+\frac{n(n-1)}{1.2}x\ldots+\frac{n(n-1)\ldots(n-2r+1)}{2r!}x^r\right\}+(1-x)^n$$
$$=\frac{(2r+1)(2r+2)\ldots(2r+n-1)}{n-1!}x^r;$$

e.g., if $n=3$, the general term is $(r+1)(2r+1)x^r$ and the quotient of

$$\frac{1+3x}{(1-x)^3}=1+6x+15x^2+\ldots,$$

putting $r=1, 2, 3$ respectively.

(2) $$\frac{(2r-1)(2r)(2r+1)\ldots(2r+n-3)}{n-1!}$$

is the general term the integral of which in finite differences is the sum of the first r coefficients in any development; if $n=4$, this sum

$$=r(r+1)(2r^2+2r-1)+6.$$

Solution by H. W. CURJEL, M.A.

(1) The $(r+1)$th term is the same as that of

$$\frac{(1+\sqrt{x})^n+(1-\sqrt{x})^n}{2(1-x)^n},\quad i.e.,\ \text{of}\ \tfrac{1}{2}\left\{(1-\sqrt{x})^{-n}+(1+\sqrt{x})^{-n}\right\};$$

and therefore $$=\frac{(n+2r-1)!}{(2r)!\,(n-1)!}x^r.$$

Putting $n=3$, we get

$$\frac{1+3x}{(1-x)^3}=1+6x+15x^2+\ldots+(r+1)(2r+1)x^r+\&c.$$

(2) Let the sum of the first r coefficients $=u_r$; then

$$\Delta u_r=\frac{(2r+1)(2r+2)(2r+3)\ldots(2r+n-1)}{(n-1)!}.$$

If $u=4$, $\quad \Delta u_r=\frac{1}{6}\left\{(2r+1)(2r+2)(2r+3)\right\}$

$$=\tfrac{1}{6}\left[(r+1)\left\{(r+2)(2r^2+6r+3)-r(2r^2+2r-1)\right\}\right];$$

therefore sum of the first r coefficients

$$=\tfrac{1}{6}\left\{r(r+1)(2r^2+2r-1)\right\}.$$

[The PROPOSER observes:—"The formula in part (2) of the question is such that, giving a value to n, each of the coefficients is obtained beginning with unity; *e.g.*, if $n=3$, the formula becomes $r(2r-1)$, and putting $r=1, 2, 3$, &c., the coefficients are 1, 6, 15, &c.; therefore it is a general term whose integral will give the summation required. The summation is otherwise obtained in the solution, but the results are the same when $n=4$ as shown."]

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14150. (R. Knowles, B.A.)—The expression $(1-10x+5x^2)+(1+x)^5$ when expanded becomes the series

$$1-\frac{5.6}{1.2}x+\ldots(-1)^r\frac{5.6\ldots(2r+3)(2r+4)}{1.2\ldots(2r-1)\,2r}x^r \text{ ad infin.}$$

It is required to deduce the expression from the series.

Solutions (1) *by* H. Forbey; (2) *by* H. W. Curjel, M.A.

$$(1)\quad \frac{1}{(1+y)^5}=(1+y)^{-5}=1-5y+\frac{5.6}{1.2}y^2-\frac{5.6.7}{1.2.3}y^3+\frac{5.6.7.8}{1.2.3.4}y^4-\ldots$$

$$=1-a_1y+a_2y^2-a_3y^3+a_4y^4-\ldots;$$

$$\frac{1}{(1-y)^5}=(1-y)^{-5}=1+a_1y+a_2y^2+\ldots.$$

Therefore $$\frac{1}{(1+y)^5}+\frac{1}{(1-y)^5}=2\left\{1+a_2y^2+a_4y^4+a_6y^6+\ldots\right\},$$

also $$=\frac{(1-y)^5+(1+y)^5}{(1-y^2)^5}=2.\frac{1+10y^2+5y^4}{(1-y^2)^5}.$$

Therefore $$\frac{1+10y^2+5y^4}{(1-y^2)^5}=1+a_2y^2+a_4y^4+\ldots.$$

Put $y=\sqrt{(-x)}$; then

$$\frac{1-10x+5x^2}{(1+x)^5}=1-a_2x+a_4x^2-a_6x^3+\ldots=1-\frac{5.6}{1.2}x+\frac{5.6.7.8}{1.2.3.4}x^2-\ldots.$$

(2) The series obviously

$$=\tfrac{1}{2}\left\{(1+i\sqrt{x})^{-5}+(1-i\sqrt{x})^{-5}\right\}=\frac{(1-i\sqrt{x})^5+(1+i\sqrt{x})^5}{2(1+x)^5}=\frac{1-10x+5x^2}{(1+x)^5}.$$

[The Proposer observes: "The expression is a particular case of the general formula in Quest. 13999 (see p. 83), from which it can be obtained by putting $n=5$, and $-x$ instead of x; the $r+1$th term in its expansion is also there given, after making the necessary change of sign.

If $$u=1-(5.6)/(1.2)\,y^2+\ldots \text{ ad infin.},$$

the solution of the symbolical equation

$$\mathrm{D}(\mathrm{D}-1)u+(\mathrm{D}+3)(\mathrm{D}+4)\,\mathrm{E}^{2\theta}u=0$$

will give the value of the series, which is thus obtained without reference to anything but the series itself. As I expected, the expression has not been deduced from the series without using impossible quantities.]

13693. (The late Professor Wolstenholme, M.A., D.Sc.)—Having given the equations

$$\frac{(a+b)\cos\theta-(a-b)\cos\phi}{a\cos\alpha}=\frac{(a+b)\sin\theta-(b-a)\sin\phi}{b\sin\alpha}=\frac{2ab}{a^2\cos^2\alpha+b^2\sin^2\alpha},$$

prove that either $\theta=\alpha$, or $\phi=\alpha$, the angles being all between 0 and 2π.

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Solution by LIONEL E. REAY, B.A.; Rev. T. MITCHESON, B.A.; *and* Professor UMES CHANDRA GHOSH.

Eliminating θ, by equating and adding values of $\cos\theta$ and $\sin\theta$, we have, after easy simplification,

$$a^2\cos^2\alpha + b^2\sin^2\alpha = ab + (a-b)(a\cos\alpha\cos\phi - b\sin\alpha\sin\phi).$$

Again, $a^2\cos^2\alpha + b^2\sin^2\alpha \equiv ab + (a-b)(a\cos^2\alpha - b\sin^2\alpha)$;

therefore $a\cos\alpha\cos\phi - b\sin\alpha\sin\phi = a\cos^2\alpha - b\sin^2\alpha$;

therefore $\sin^2\phi\,(a^2\cos^2\alpha + b^2\sin^2\alpha) + 2b\sin\alpha\,(a\cos^2\alpha - b\sin^2\alpha)\sin\phi + (a\cos^2\alpha - b\sin^2\alpha)^2 - a^2\cos^2\alpha = 0$;

therefore values of $\sin\phi$ are $\sin\alpha$, or $\dfrac{(a\cos^2\alpha - b\sin^2\alpha)^2 - a^2\cos^2\alpha}{(a^2\cos^2\alpha + b^2\sin^2\alpha)\sin\alpha}$. Substituting this value of $\sin\phi$ in the original equation

$$(a+b)\sin\theta = \frac{(b-a)\left[(a\cos^2\alpha - b\sin^2\alpha)^2 - a^2\cos^2\alpha\right] + 2ab^2\sin^2\alpha}{(a^2\cos^2\alpha + b^2\sin^2\alpha)\sin\alpha},$$

which reduces to $\sin\theta = \sin\alpha$; therefore either

$$\sin\phi = \sin\alpha \quad \text{or} \quad \sin\theta = \sin\alpha.$$

Similarly $\cos\phi = \cos\alpha$ or $\cos\theta = \cos\alpha$;

therefore α equals either ϕ or θ, the angles being between 0 and 2π.

13946. (I. ARNOLD.)—Given two points A and B, one at the foot of a slope and the other at the top; the distance between these points is a miles, and the ascent of the connecting line is 1 foot in n feet; required the length, l, of a road from A to the vertical line through B that it may have an ascent of 1 foot in $(m+n)$ feet.

Solution by H. W. CURJEL, M.A.; H. J. WOODALL, A.R.C.S.; *and others.*

Distance between A and B measured horizontally $= a\,(1 - 1/n^2)^{\frac{1}{2}}$;

therefore $l^2 = a^2\,(1 - 1/n^2) + [l/(m+n)]^2$;

therefore $l = n^{-1}\,[a\,(m+n)]\,\{(n^2-1)/[(m+n)^2-1]\}^{\frac{1}{2}}$.

13992. (R. TUCKER, M.A.)—Points D, E, F; D′, E′, F′ are taken on the sides of a triangle ABC, and perpendiculars are let fall from them on the sides. The feet of the respective pairs of perpendiculars are joined, thus forming six triangles. If these triangles are all equal, then the points D, E, F; D′, E′, F′ lie on a circle concentric with the circle ABC.

Solution by LIONEL E. REAY, B.A.

Let D be point $(0\beta_1\gamma_1)$ and D′ $(0\beta_1'\gamma_1')$,

E ,, $(\alpha_2 0\gamma_2)$ and E′ $(\alpha_2'0\gamma_2')$, &c.

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Then twice the area of each triangle

$= \beta_1\gamma_1 \sin A = \gamma_2\alpha_2 \sin B = \alpha_3\beta_3 \sin C = \beta_1'\gamma_1' \sin A = \ldots = k$, a constant.

The points lie on the curve whose equation is

$$\beta\gamma \sin A + \gamma\alpha \sin \beta + \alpha\beta \sin C = k,$$

that is, a circle concentric with ABC.

13965. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize $(722^{10}+1)$.

Solution by the PROPOSER.

Here $N = 1^4 + 2^2(2^2 \cdot 19^5)^4$, and contains the factor

$$N_1 = (722^2+1) = 1^4 + 2^2 \cdot 19^4.$$

Now N, N_1 are both of form (x^4+4y^4) which is always resolvable into the two factors L, M, where

$$L = x^2 - 2xy + 2y^2, \quad M = x^2 + 2xy + 2y^2.$$

Hence, if $N = LM$ and $N_1 = L_1M_1$, then

$$L = 1 - 2^3 \cdot 19^5 + 2^5 \cdot 19^{10}, \qquad M = 1 + 2^3 \cdot 19^5 + 2^5 \cdot 19^{10},$$
$$L_1 = 1 - 2 \cdot 19 + 2 \cdot 19^2 = 5 \cdot 137, \quad M_1 = 1 + 2 \cdot 19 + 2 \cdot 19^2 = 761;$$

and in such numbers L contains L_1 or M_1, whilst M contains M_1 or L_1. Calling the quotients S, T, division gives

$$S = L \div M_1 = 257{,}810{,}907{,}281; \quad T = M \div L_1 = 286{,}414{,}803{,}005.$$

Next, REUSCHLE'S *Tafeln complexer Primzahlen*, Berlin, 1875, p. 471, gives 821 and 881 as factors of N; casting these out gives

$$S = 881 \cdot 292{,}634{,}401; \quad T = 5 \cdot 821 \cdot 69{,}772{,}181.$$

A MS. table showing value of a in numbers of form $(a^{10}+1) \div (a^2+1)$, compiled by the present writer, shows that N has no more divisors $< 10{,}000$. Applying now the method of factorization given by Mr. F. W. LAWRENCE (*Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 465), one finds*

$$S = 881 \cdot 15{,}401 \cdot 19{,}001; \quad 69{,}772{,}181 = \textit{prime}.$$

Therefore $N = (5 \cdot 137)(761)(881 \cdot 15{,}401 \cdot 19{,}001)(5 \cdot 821 \cdot 69{,}772{,}181)$.

14152. (DISCIPULUS.)—An alphabet consists of m consonants and n vowels, m being greater than n. Find the number of ways in which all the letters may be arranged so that each of the vowels shall be between two consonants.

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* The detail of this could not be given in any short compass.

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Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and* Rev. W. A. WHITWORTH, M.A.

For any *particular* arrangement of the m consonants there are $(m-1)$ *spaces* between them available for one vowel each. Any particular vowel can be inserted in one of those spaces in $(m-1)$ ways; a second vowel can be placed in one of the $(m-2)$ *remaining* spaces in $(m-2)$ ways; a third vowel in $(m-3)$ ways; and so on; and, lastly, there will be $(m-n)$ spaces remaining available for the nth vowel, which can therefore be introduced in $(m-n)$ ways. Thus *all* the n vowels can be introduced at once in $(m-1)(m-2)(m-3)\ldots(m-n) = (m-1)!\div(m-n-1)!$ ways.

This exhausts the ways of *placing* and also of *permuting* the vowels. The m consonants of the above arrangement can be permuted in $m!$ ways. Hence the total number of arrangements required is

$$m!\,(m-1)!\div(m-n-1)!.$$

14026. (R. F. DAVIS, M.A.)—If a, b, c, d be the sides of a cyclic quadrilateral, and $2s = a+b+c+d$, the angle ϕ between its diagonals is given by the equation $\tan\frac{1}{2}\phi = \sqrt{\left\{\frac{(s-a)(s-c)}{(s-b)(s-d)}\right\}}$.

Solution by Rev. T. MITCHESON, B.A.; C. JOSS, M.A.; *and many others.*

Since $$\tan^2\tfrac{1}{2}\phi = \frac{(s-a)(s-c)}{(s-b)(s-d)},$$

then $$\frac{1+\tan^2\frac{1}{2}\phi}{1-\tan^2\frac{1}{2}\phi} = \frac{(s-a)(s-c)+(s-b)(s-d)}{(s-b)(s-d)-(s-a)(s-c)},$$

i.e., $$\frac{1}{\cos\phi} = \frac{2(ac+bd)}{a^2+c^2-b^2-d^2}.$$

Now it is a well known fact that AC.BD $= ac+bd$. Let p, q, r, and s be the respective segments made by the intersection of AC and BC. Then

$$\cos\phi = \frac{p^2+q^2-a^2}{2pq} = \frac{r^2+s^2-c^2}{2rs} = \frac{b^2-q^2-r^2}{2qr} = \frac{d^2-p^2-s^2}{2ps},$$

or $$2\cos\phi\,(pq+ps+qr+rs) = b^2+d^2-a^2-c^2,$$

and $$\cos\phi = \frac{b^2+d^2-a^2-c^2}{2AC.BD} = \frac{b^2+d^2-a^2-c^2}{2(ac+bd)};$$

therefore, &c.

14078. (Professor UMES CHANDRA GHOSH.)—Show that

$\Gamma\left\{\frac{1}{2}(2m+1)\right\} = (2m-1)(2m-3)(2m-5)\ldots5.3.1.(\sqrt{\pi})/2^m$,

where m is any integer. Hence show that

$$\Gamma(\tfrac{1}{2}) + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} + \frac{\Gamma(\frac{5}{2})}{\Gamma(3)} + \frac{\Gamma(\frac{7}{2})}{\Gamma(4)} + \&c.\text{ to }\infty = 0. \quad [?]$$

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Remarks by H. A. WEBB.

To prove that $\Gamma\left\{\frac{1}{2}(2n+1)\right\} = (1/2^n)\,1.3.5\ldots(2n-1)\,\sqrt{\pi}$,

and hence that $\Gamma(\frac{1}{2}) + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} + \frac{\Gamma(\frac{5}{2})}{\Gamma(3)} + \ldots = 0$ (?).

[This series is divergent.]

Since $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

we have
$$\begin{aligned}\Gamma\left\{\tfrac{1}{2}(2n+1)\right\} &= \tfrac{1}{2}(2n-1)\,\Gamma\left\{\tfrac{1}{2}(2n-1)\right\} = \ldots \\ &= (1/2^n)\,1.3.5\ldots(2n-1)\,\Gamma(\tfrac{1}{2}) \\ &= (\sqrt{\pi}/2^n)\,1.3.5\ldots(2n-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).\end{aligned}$$

Also $\Gamma(n+1) = n!$, when n is integral. Therefore

$$\begin{aligned}&\Gamma(\tfrac{1}{2}) + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} + \frac{\Gamma(\frac{5}{2})}{\Gamma(3)} + \ldots \\ &\quad = \sqrt{\pi}\left\{1 + \frac{1}{2}\cdot\frac{1}{1!} + \frac{1.3}{2^2}\cdot\frac{1}{2!} + \frac{1.3.5}{2^3}\cdot\frac{1}{3!} + \ldots\right\} \\ &\quad = \sqrt{\pi}\left\{1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),\end{aligned}$$

a *divergent* series, for $\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} > \frac{1}{4n}\cdot\frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)}$,

and thus each term in the bracket $> (1/\sqrt{2})$.

We have, however, that

$$\Gamma(\tfrac{1}{2}) - \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} + \frac{\Gamma(\frac{5}{4})}{\Gamma(3)} - \ldots = \sqrt{\pi}\,(1+1)^{-\frac{1}{2}} = \sqrt{(\tfrac{1}{2}\pi)} \quad \ldots\ldots\ldots (3),$$

[Professor SANJÁNA bears out the above criticism as follows:—

$$\begin{aligned}\Gamma\left\{\tfrac{1}{2}(2m+1)\right\} &= \Gamma\left\{\tfrac{1}{2}(2m-1)\right\}\left\{\tfrac{1}{2}(2m-1)\right\} \\ &= \Gamma\left\{\tfrac{1}{2}(2m-3)\right\}\left\{\tfrac{1}{2}(2m-3)\right\}\left\{\tfrac{1}{2}(2m-1)\right\} \\ &= \Gamma(\tfrac{1}{2})\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots\tfrac{1}{2}(2m-1) = \sqrt{\pi}.1.3.5\ldots(2m-1)/2^m.\end{aligned}$$

Hence the given series

$$= \sqrt{\pi} + \tfrac{1}{2}\cdot\frac{\sqrt{\pi}}{1} + \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\frac{\sqrt{\pi}}{2!} + \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\frac{\sqrt{\pi}}{3!}\ldots = \sqrt{\pi}\,(1-1)^{-\frac{1}{2}} = \infty.$$

If the signs are alternately + and −, the result will be

$$\sqrt{\pi}\,(1+1)^{-\frac{1}{2}} = \sqrt{(\tfrac{1}{2}\pi)}.$$

In fact the given series is seen to be

$$\int_0^\infty e^{-x}x^{-\frac{1}{2}}\left(1 + \frac{x}{1} + \frac{x^2}{2!}\ldots \text{ad inf.}\right)dx = \int_0^\infty x^{-\frac{1}{2}}dx = \infty;$$

the other series is

$$\int_0^\infty e^{-x}x^{-\frac{1}{2}}(e^{-x})\,dx = \int_0^\infty e^{-2x}x^{-\frac{1}{2}}dx = \frac{\sqrt{\pi}}{\sqrt{2}}.$$

In neither case can the result be zero.]

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14175. (R. TUCKER, M.A.)—The normal at P, on a parabola, meets the directrix at Q. Show that QS meets (1) the tangent at P, (2) the diameter through P, in cubic curves.

Solution by K. G. PANCHAPAGESA AIYAR; Rev. T. MITCHESON, B.A.; *and the* PROPOSER.

Let the coordinates of P be $(am^2, -2am)$, so that the equation of the normal at P is $y = mx - 2am - am^3$ ..... (i.).

The equation of the directrix is $x = -a$ ..... (ii.).

Therefore, from (i.) and (ii.), Q is a point $[-a, -(3am + am^3)]$.

Again, the equation of QS is $\frac{x+a}{a+a} = \frac{y+(3am+am^3)}{3am+am^3}$;

i.e., $x(3m+m^3) - 2y - 3am - am^3 = 0$ ..... (iii).

And the equation of the tangent at P to the parabola is

$$am^2 + my + x = 0 \text{ ..... (iv.).}$$

Now, solving (iii.) and (iv.) by the rule of cross-multiplication, we have

$$x = \frac{am^4 + am^2}{m^4 + 3m^2 + 2} = \frac{am^2}{m^2+2} \quad \text{and} \quad y = -\frac{am(m^2+3)}{m^2+2};$$

therefore $y = -\left[\frac{am^3}{m^2+2} + \frac{3am}{m^2+2}\right] = -\left[mx + \frac{3x}{m}\right]$;

i.e., $m^2x + my + 3x = 0$ ..... (v.).

Again, solving (iv.) and (v.) by the rule of cross-multiplication for m^2 and m, we have $m^2 = -\frac{2xy}{xy-ay}$ and $m = \frac{3ax-x^2}{xy-ay}$;

therefore $-\frac{2xy}{xy-ay} = \frac{x^2(3a-x)^2}{(xy-ay)^2}$;

i.e., $x^3 + 2xy^2 - 6ax^2 - 2ay^2 + 9a^2x = 0$,

which is the locus required, and, since this equation is of the third degree, the locus is a cubic curve.

(2) The equation to QS is given by (iii.) above, and the equation to the diameter through P is $y = -2am$ ..... (vi.).

If we eliminate m between (iii.) and (vi.), we get

$$(x-a)(2a^2+y^2) + 16a^3 = 0,$$

which is the locus required. The locus is a cubic curve, since its equation is of the third degree.

14164. (Professor ELLIOTT, F.R.S.)—If a, b are positive moduli, and $\alpha - \beta$ lies between 0 and 2π, and if $f(z)$ is a function of the complex variable $z = re^{i\theta}$ which approaches definite limits $f(0)$ and $f(\infty)$ as z

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comes infinitely small and infinitely great within the sector from $\theta = \beta$ to $\theta = \alpha$, and is holomorphic within and on the bounding lines of the sector except for simple poles at $c_1, c_2, \ldots, c_n$ within the sector, prove the FRULLANIAN formula

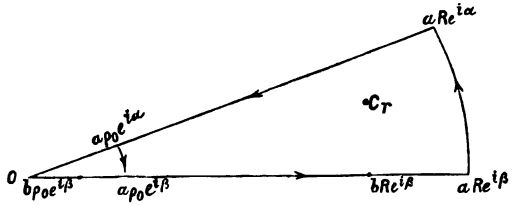
$$\int_0^\infty \frac{f(ae^{i\alpha}x) - f(be^{i\beta}x)}{x}\,dx$$
$$= \{f(\infty) - f(0)\}\{\log a/b + i(\alpha - \beta)\} - 2\pi i \sum_{r=1}^{r=n} \left[(z-c_r)\frac{f(z)}{z}\right]_{z=c_r},$$

the variable x being real.

Solutions (1) *by* G. H. HARDY, B.A.; (2) *by* H. W. CURJEL, M.A.

(1) Integrate $\int \frac{f(z)}{z}\,dz$ round the contour bounded by $\theta = \alpha$, $\theta = \beta$, $r = a\rho_0$, $r = a\mathrm{R}$; where ρ_0 decreases, and R increases beyond limit.

$$\{f(\infty) - f(0)\}\int_\beta^\alpha i\,d\theta + \int_{b\rho_0 e^{i\beta}}^{b\mathrm{R}e^{i\beta}} \frac{f(z)}{z}\,dz - \int_{a\rho_0 e^{i\alpha}}^{a\mathrm{R}e^{i\alpha}} \frac{f(z)}{z}\,dz$$
$$- f(0)\int_{b\rho_0 e^{i\beta}}^{a\rho_0 e^{i\beta}} \frac{dz}{z} + f(\infty)\int_{b\mathrm{R}e^{i\beta}}^{a\mathrm{R}e^{i\beta}} \frac{dz}{z}$$
$$= 2\pi i \sum_{r=1}^{r=n} \left[(z-c_r)\frac{f(z)}{z}\right]_{z=c_r},$$



which reduces to

$$\int_0^\infty \frac{f(ae^{i\alpha}x) - f(be^{i\beta}x)}{x}\,dx = \{f(\infty) - f(0)\}\{i(\alpha-\beta) + \log a/b\}$$
$$- 2\pi i \sum_{r=1}^{r=n} \left[(z-c_r)\frac{f(z)}{z}\right]_{z=c_r},$$

x being real.

(2) Take points A′A, B′B on the radii vectores bounding the sector, and let OA′ : OB′ = OA : OB = a : b, OA′ being very small, and OA very great. Take the integral $\int \frac{f(z)}{z}\,dz$ round the contour B′BAA′B′ which should be made to include all the poles $c_1, c_2, \ldots, c_n$. The integral

$$= 2\pi i \sum_{r=1}^{r=n} \left[(z-c_r)\frac{f(z)}{z}\right]_{z=}.$$

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Integral along B′B + integral along AA′ becomes when OA, OB are increased and OA′, OB′ diminished indefinitely

$$= \int_0^\infty \frac{-f(ae^{i\alpha}x) + f(be^{i\beta}x)}{x}\,dx.$$

Integral along $BA = f(\infty)\int_B^A \frac{dz}{z} = f(\infty)\left\{\log\frac{a}{b} + i(\alpha-\beta)\right\}$.

Integral along $A'B' = -f(0)\int_{B'}^{A'}\frac{dz}{z} = -f(0)\left\{\log\frac{a}{b} + i(\alpha-\beta)\right\}$.

Hence the required result.

14083. (C. E. Hillyer, M.A.)—Show that the collinearity of the mid-points of the diagonals of a complete quadrilateral is an immediate deduction from Menelaus' theorem, viz., that the ratio compounded of the ratios of the segments into which a straight line divides the sides of a triangle is -1, and its converse.

Solution by the Proposer *and* Prof. N. Bhattacharyya.

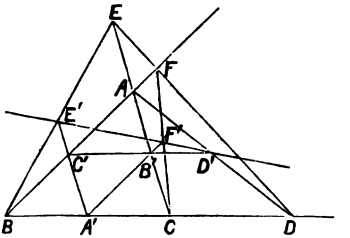
Let a straight line meet BC in D, CA in E, and AB in F. Then, if D′ is the mid-point of AD, B′ of AC, C′ of AB, we have C′, B′, D′ in a straight line and parallel to BC, and

$$C'D' : D'B' = BD : DC.$$

Similarly, E′ is a point in C′A′ such that

$$A'E' : E'C' = CE : EA;$$

and F′ in A′B′ such that



$$B'F' : F'A' = AF : FB.$$

But $$BD \cdot CE \cdot AF : DC \cdot EA \cdot FB = -1.$$

Therefore $$C'D' \cdot A'E' \cdot B'F' : D'B' \cdot E'C' \cdot F'A' = -1.$$

Therefore D′, E′, F′ are collinear; and these are the mid-points of the diagonals of the quadrilateral ACDF.

14190. (C. E. Hillyer, M.A.)—If P be a point on the circumcircle of ABC, whose orthocentre is H, and a be the orthocentre of the triangle PBC, b of the triangle PCA, and c of the triangle PAB; prove (1) that B, a, H, C are concyclic; (2) that the Simson-line of P with respect to the triangle ABC is also the Simson-line of a with respect to the triangle HBC, and of b with respect to HCA, and of c with respect to HAB; (3) that this line bisects Aa, Bb, Cc, and PH.

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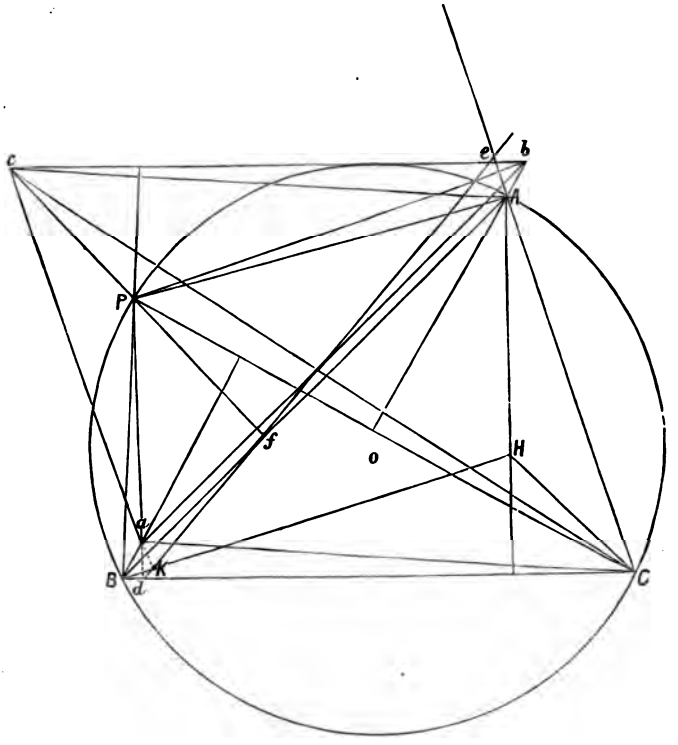
Solution by R. TUCKER, M.A.; W. H. SALMON, B.A.; *and* K. G. PANCHAPAGASA AIYAR.

(1) Because angles B*a*C, BHC are supplements of the equal angles P, A; therefore, &c.

(2) $\angle BKd = BPf - HBC = C - PBA = PCB = \angle Bad$;
therefore B, *a*, K, *d* are concyclic, and $\angle BKa$ a right angle; therefore, &c.



(3) P*a*, AH each = 2 (perpendicular from O on BC), and parallel to one another. Therefore AH*a*P is a parallelogram, and A*a* bisects HP, and so for the others; and, by a well-known property, the SIMSON-line *dfe* bisects HP.

(4) Again $aH = AP$ and parallel to it } therefore $\angle aHb = 180° - C$.
$bH = BP$ and parallel to it } therefore *abc*H concyclic.

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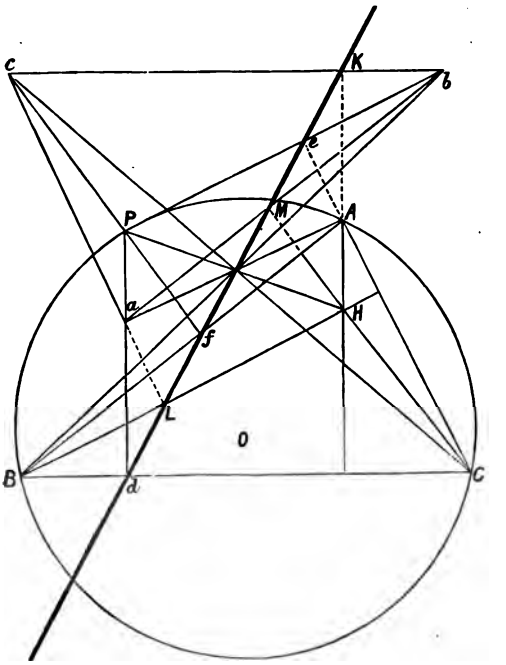
(5) The loci of a, b, c are evidently circles, as P moves round the circumference.

(6) $$\angle aPb = \angle APB = \angle AHB.$$

Therefore, by (3), $\triangle abc$ is congruent with $\triangle ABC$, and P is its orthocentre; its sides are also (because Ba is equal and parallel to Ab) parallel to the sides of ABC. Also, from (3), the mid-point of HP is the centre of perspective of the triangles.

[The PROPOSER observes: "It is not necessary to assume that the SIMSON-line dfe bisects HP; for, if dfe meet AH in K, then bK is perpendicular to AH, and therefore parallel to Bd, AK is parallel to ad, and Ab is both equal and parallel to Ba, each being double of the perpendicular from O on PC. Therefore AK is equal as well as parallel to ad, and therefore dK passes through the middle point of Aa, which is also the middle point of HP.



"The question is the outcome of an attempt to find a simpler proof of this 'well-known property' than that usually given.

"It is to be noticed that the intersections of AH, BH, CH with the line dfe are on the lines bc, ca, ab respectively."].

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13977. (Rev. T. MITCHESON, B.A.)—If (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the three tangential points on a parabola from which normals can be drawn to a given point, find the length of the ordinate of the co-normal point.

Solution by the PROPOSER; R. KNOWLES, B.A.; *and* G. W. PRESTON, B.A.

Let (β, γ) be the co-normal point, and the equation to the normal through that point is $2a\gamma - 2ay_1 = \beta y_1 + x_1 y_1$,

since $y_1^2 = 4ax_1$; hence, for finding y_1, we have

$$y_1^3 + 4a(2a-\beta)y_1 - 8a^2\gamma = 0;$$

therefore $y_1 + y_2 + y_3 = 0$, $y_1y_2 + y_1y_3 + y_2y_3 = 4a(2a-\beta)$,

and $y_1y_2y_3 = 8a^2\gamma$.

From these equations we get

$4ax_1 + 4ax_2 + 4ax_3 = 8a(2a-\beta)$ and $1/y_1 + 1/y_2 + 1/y_3 = (2a-\beta)/2a\gamma$,

and therefore the ordinate required is

$$\tfrac{1}{4}(x_1 + x_2 + x_3)(y_1^{-1} + y_2^{-1} + y_3^{-1}).$$

[Mr. KNOWLES remarks:—"The ordinate required is γ, and in the solution it is shown that $\gamma = y_1y_2y_3/8a^2$."]

14135. (Professor THOMAS SAVAGE.)—For what positive integral value or values of n is $(n+1)^n - n^n$ divisible by $2mn+1$, where m is any positive integer?

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

$\{(n+1)^n - n^n\}$ is always divisible by *some* prime p of form $p = 2mn+1$, where m is *some* positive integer (not *any* integer as stated in the question). n may be *any* positive integer.

[This is believed to have been proved by EULER, for the more general case $(a^n - b^n)$ is always divisible by $p = 2mn+1$ = prime.

The PROPOSER observes: "Owing to an error—unfortunately not detected at first revision of manuscripts—in the working of a particular case of FERMAT's theorem ($x^n + y^n \neq z^n$, for integers), I was led, by the result, to set the question as printed.]

14242. (C. JOSS, M.A.)—Tangents to a circle at the vertices of an inscribed triangle meet the opposite sides produced in collinear points.

Solution by Rev. J. CULLEN *and* Prof. N. BHATTACHARYYA.

By applying PASCAL's theorem to the hexagon inscribed in a TUCKER circle we see that the anti-parallels FE′, F′D, and D′E intersect the

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sides in collinear points. When the TUCKER circle becomes the circumcircle the PASCAL line becomes LEMOINE's line. (See figure in Quest. 14159.)

[The PROPOSER observes: "This results from PASCAL's theorem by diminishing the alternate sides of the hexagon until the secants become tangents. The quadrilateral and pentagon are likewise included in the general form:—If six lines cut one another in concyclic points, lines 1 and 4, 2 and 5, 3 and 6 meet in collinear points. See McDOWELL's *Exercises* (1863), No. 224."]

14257. (Professor COCHEZ.)—Etudier $x^4 + y^4 + 3x^2y + y^2 = 0$.

Solution by H. J. WOODALL, A.R.C.S.

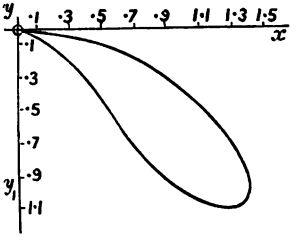
By solving the equation, we find

$$x^2 = -\tfrac{3}{2}y \pm \tfrac{1}{2}y\,(5 - 4y^2)^{\frac{1}{2}},$$

whence y must be negative and not greater than $-\frac{1}{2}\sqrt{5}$, *i.e.*, $-1{\cdot}118....$ Also the curve is symmetrical with respect to the axis of y, $y = -1$ cuts the curve where $x^4 - 3x^2 + 2 = 0$; therefore the points $(\pm\sqrt{2}, -1), (\pm\sqrt{1}, -1)$ are on the curve. The curve is parallel to the x-axis where $du/dx = 0$, *i.e.*, $4x^3 + 6xy = 0$; this gives the points $(0,0), (\pm 1{\cdot}295..., -1{\cdot}118...)$. Properly, the curve at the point $(0, 0)$ requires further examination, since here

$$du/dy = 0$$

also. The x-axis is, however, a double tangent at $(0, 0)$. The curve is parallel to the y-axis at $(\pm 1{\cdot}414..., -1)$, and one-half of it is represented in the accompanying figure.



14275. (R. J. DALLAS.)—Being given four points A, B, C, D, prove that the nine-point circles of the four triangles formed by leaving out one point in turn all pass through one point.

Solution by Rev. T. ROACH, M.A.; W. C. STANHAM, B.A.; *and others.*

The nine-point circle of any triangle is the locus of the centre of equilateral hyperbolas described about the triangle. Let one of the hyperbolas be described about ABC and pass through D. Then the centre of the equilateral hyperbola through ABCD lies on the nine-point circle of ABC. Similarly, it lies on the nine-point circles of BCD, CDA, DAB; therefore these four circles all pass through one point.

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14291. (R. F. Davis, M.A.)—If ABCD be any quadrilateral circumscribed to a circle (centre O), then $AB.AD/OA^2 = CB.CD/OC^2$.

Solution by C. E. McVicker, M.A.

$\alpha+\beta+\gamma+\delta = \pi$; therefore AOD can be divided into β and γ, also BOC can be divided into α and δ. By similar triangles,

$$OA^2 = AB.AX;$$

therefore $OA^2/(AB.AD) = AX/AD$;

similarly, $OC^2/(CB.CD) = CY/CB$.

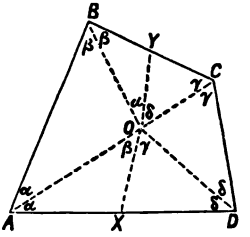
It is now required to show that XY divides the sides proportionally.

$AX/XD = (AX/XO)(XO/XD)$

$= (\sin\beta/\sin\alpha)(\sin\delta/\sin\gamma)$,

$CY/YB = (\sin\delta/\sin\gamma)(\sin\beta/\sin\alpha)$,

which is the same; whence the theorem follows at once.



14276. (R. Knowles, B.A.)—At a point R (eccentric angle θ) of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, chords RP, RQ are drawn at right angles; the normal at R meets PQ in M and the curve in N. Prove (1) that $RM : MN = a^4\sin^2\theta + b^4\cos^2\theta : a^2b^2$. (2) If R $(am^2, 2am)$ be on the parabola $y^2 = 4ax$, prove that $RM = m^2MN$.

Solution by G. W. Preston, B.A.

Draw the double ordinate RHQ, and draw MF, NG perpendicular to CA. Now M is the Frégier point of the point R; and therefore CMQ is a straight line and $CQ : CM = a^2 + b^2 : a^2 - b^2$. (*Vide* Russell's *Geometry*, p. 201, Ex. 6.)

But, by parallels, $CQ : CM = CH : CF$;

$\therefore$ $CH : CF = a^2 + b^2 : a^2 - b^2$; $\therefore$ $CH : FH = a^2 + b^2 : 2b^2$;

$\therefore$ $CF = \{(a^2-b^2)/(a^2+b^2)\}(a\cos\theta)$ and $FH = \{2b^2/(a^2+b^2)\}(a\cos\theta)$.

It is clear, from the figure, that $RM : MN = FH : FG$.

Now

$FG = CF - CG$

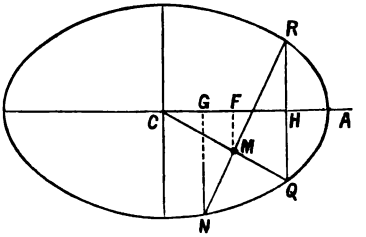
$= CF - (CG + CH) + CH$.

Equation of normal at R is

$a\sin\theta.x - b\cos\theta.y$

$= (a^2-b^2)\sin\theta\cos\theta$.

Substituting the value of y from this, in the equation of curve, we get



$$b^2x^2 + \frac{a^2\{a\sin\theta.x - (a^2-b^2)\sin\theta\cos\theta\}^2}{b^2\cos^2\theta} = a^2b^2,$$

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a quadratic in x, the sum of whose roots is

$$CG + CH = \frac{2a^3(a^2-b^2)\sin^2\theta\cos\theta}{a^4\sin^2\theta + b^4\cos^2\theta};$$

$$\therefore\quad FG = \frac{a^2-b^2}{a^2+b^2}(a\cos\theta) - \frac{2a^3(a^2-b^2)\sin^2\theta\cos\theta}{A} + a\cos\theta$$

(where $A = a^4\sin^2\theta + \&c.$)

$$= \frac{a\cos\theta}{A(a^2+b^2)}\left\{2a^2A - 2a^2(a^4-b^4)\sin^2\theta\right\}$$

$$= \frac{a\cos\theta}{A(a^2+b^2)}\left\{2a^6\sin\theta + 2a^2b^4\cos^2\theta - 2a^6\sin^2\theta + 2a^2b^4\sin^2\theta\right\}$$

$$= \frac{a\cos\theta}{A(a^2+b^2)}\,2a^2b^4;$$

$$\therefore\quad \frac{FH}{FG} = \frac{2b^2}{a^2+b^2}(a\cos\theta)\,\frac{A(a^4+b^2)}{a\cos b\,.\,2a^2b^4} = \frac{A}{a^2b^2};$$

therefore $$RM : MN = a^4\sin^2\theta + b^4\cos^2\theta : a^2b^2.$$

In the parabola, RM is bisected by the axis. (*Vide* RUSSELL'S *Geometry*, p. 201, Ex. 3.)

Draw the ordinate NP, and draw RQ, MT parallel to the axis to meet NP at Q and T.

The equation of the normal at R is

$$(y-2am)/(x-am^2) = -m$$

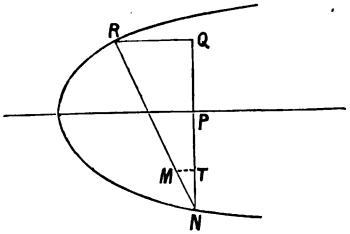
or $$mx + y - \&c. = 0.$$

Put $$x = y^2/4a;$$

$$\therefore\quad my^2 + 4ay - \&c. = 0;$$

therefore sum of ordinates

$$= -4a/m,$$



and, since they are of opposite signs and RM is bisected, therefore

$$NT = -4a/m.$$

Now $$RM : MN = QT : NT = 4am : 4a/m = m^2 : 1;$$

therefore $$RM = m^2\,.\,Mn.$$

14094. (ROBERT W. D. CHRISTIE.)—In my factorization table (*vide* Vol. LXIX., p. 99) prove the law $a = 10t + u$, $b = Mu + t$, where t signifies the tens digit, u the units, and M the multiplier.

E.g.—Under prime 79 we have

$15 = 10\times1+5$ and $41 = 8\times5+1$, $12 = 8\times1+4$, $17 = 8\times2+1$, &c.,

where $1501 = 41001 = 120001 = 1700001 = 79$ (M), &c., *ad lib.*

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Solution by the PROPOSER.

Let P be any number 10M−1 prime to 10, where M signifies the multiplier, and let $a_1, a_2, a_3, \ldots, a_n$ represent the digits in the vertical columns of the table whose general law is $a_n 10^n + 1 = 0 \bmod P$. We have to prove that $Mu + t = a_{n+1}$, where $a_n = 10t + u$.

By the general law of the table,

$$a_n 10^n + 1 = 0 \bmod P \text{ and } a_{n+1} 10^{n+1} + 1 = 0 \bmod P;$$

therefore $$10^n (10a_{n+1} - a_n) = 0 \bmod P;$$

i.e., $$10a_{n+1} - a_n = 0 \bmod P \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

since 10 is prime to P by hypothesis.

Again, by (1), $10(Mu + t) - (10t + u) = 0 \bmod P$;

therefore $u(10M - 1) = 0 \bmod P$, *i.e.*, $10M - 1 = 0 \bmod P$,

when M is the multiplier.

14147. (Rev. W. ALLEN WHITWORTH, M.A.) — If three square numbers (not having a common factor) be in arithmetical progression, the middle one divided by 120 will leave a remainder 1 or 25 or 49, and each of the others divided by 240 will leave a remainder 1 or 49. For example: 49, 289, 529; 289, 625, 961; 961, 1681, 2401.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and* R. W. D. CHRISTIE.

Let x^2, y^2, z^2 be the three squares, such that $x^2 + z^2 = 2y^2$ $(x < y < z)$. The general solution of this has been shown in a recent number of this journal to be

$$x = (m \sim 2n)^2 - 2n^2, \quad y = (m \sim n)^2 + n^2, \quad z = m^2 \sim 2n^2.$$

Hence x, y, z will be *all even* if m is *even* (a case excluded since x, y, z are to have no common factor); and x, y, z will be *all odd* if m be *odd*. Thus x, y, z, m must be *all odd*. Now *all odd squares* are of form $(8\alpha + 1)$; and *all squares* are of *one* of the forms 3β, $(3\beta + 1)$, and also of *one* of the forms $(5\gamma - 1)$, 5γ, $(5\gamma + 1)$. Now, since y is of form $(a^2 + b^2)$, *all* its factors must be of that form, so y cannot have the factor 3, and y^2 cannot be of form 3β. Similarly, x^2 and z^2 are both of form $(c^2 \sim 2f^2)$, x and z are each of one of the forms $(8\alpha \pm 1)$, and all their factors must be of form $(c^2 \sim 2f^2)$; hence x^2 and z^2 are both of form $(16\alpha + 1)$; and x and z cannot have the factors 3 or 5; so that x^2 and z^2 cannot be of form 3β or 5γ.

Hence $$y^2 = 8\alpha + 1 = 3\beta + 1 = 5\gamma - 1, \text{ or } 5\gamma, \text{ or } 5\gamma + 1;$$

$$x^2 \text{ and } z^2 = 16\alpha + 1 = 3\beta + 1 = 5\gamma - 1 \text{ or } 5\gamma + 1.$$

Compounding the three forms,

$$y^2 = 120\eta + 1, 25, \text{ or } 49; \quad x^2 \text{ and } z^2 = 240\xi + 1 \text{ or } 49.$$

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14088. (Rev. T. Roach, M.A.)—OP and OQ are tangents to a parabola: prove $OP^2 : OQ^2 = SP : SQ$.

Solution by H. A. Webb; *the* Proposer; *and many others.*

Draw pp', qq' parallel to OP, OQ respectively.

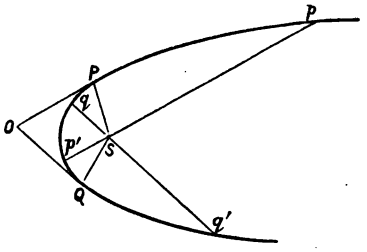
pp', qq' are the parameters of the diameters through P, Q, and are equal to 4SP, 4SQ respectively.

Also $\frac{Sp.Sp'}{pp'} = \frac{Sq.Sq'}{qq'}$ = half the semi-latus rectum.

Then, by Boscovitch's theorem,

$OP^2 : OQ^2 = Sp.Sp' : Sq.Sq'$

$= pp' : qq' = SP : SQ$.



[The Proposer observes that the theorem follows directly from Drew's Prop. xvii. regarding intersecting chords, when considered as the limit.]

14254. (C. E. Bickmore, M.A.)—Is the number 78,875,943,472,201 prime or composite? It is a factor of $(10^{25}+1) \div (10^5+1)$; has no factor between 1 and 100,000; and any factor when divided by 200 leaves one of the two remainders 1 or 51.

Solution by Rev. J. Cullen, B.A.

Adopting Mr. Biddle's method and notation as given in his note to Mr. Lawrence's paper on the "Determination of certain Primes" (see Vol. lxxi., p. 113), we have

$$\Delta = 25, \quad M = 3,155,037,738,888 = \Delta R + a;$$

therefore $R = 126,201,509,555$ and $a = 13$.

$$R \equiv 3 \pmod 4, \quad PQ \equiv 0 \pmod 4 = R - r;$$

therefore $r \equiv 3 \pmod 4$, $H = \frac{1}{2}\Delta^2 r + \frac{1}{2}\Delta a + 1 = 1250m + 1101$.

Furthermore we have that $P = 8p$ or $8p+2$ and $Q = 8q$ or $8q+2$, but the form of H shows that we must take $P = 8p$ and $Q = 8q$ together, or else $P = 8p+2$ and $Q = 8p+2$; as otherwise we should have

$$H = \tfrac{1}{2}\{200(p+q) + 52\},$$

which cannot end in 1. Therefore $P - Q = 8(p-q)$, and there are two cases to consider.

First Case.— $PQ = 64pq$, $R \equiv 51 \pmod{64} \equiv r$.

Hence $H_m = 20,000m + 16,101$.

Second Case.—

$$PQ = 64pq + 16(p+q) + 4 \quad \text{and} \quad H = 100(p+q) + 51;$$

therefore $40,000pq + 10,200(p+q) + 2601 = N = \ldots 72,201$;

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therefore $p+q$ must end in 3 or 8. Taking 3, we have

$$\ldots i3 \,.\, 10{,}200 + 2601 = \ldots (2i+3)\, 201.$$

Now, i being any integer, $(2i+3)$ cannot end in 2. Hence $p+q$ must end in 8. Again, $\ldots i8 \,.\, 10{,}200 + 2601 = \ldots (2i+4)\, 201$;
therefore i must end in 4 or 9. Taking 9,

$$\ldots i98 \,.\, 10{,}200 + 2601 = \ldots (2i+10)\, 2201;$$

but $(2i+10)$ + a multiple of 4 cannot end in 7. Thus

$$p+q \equiv 48 \pmod{100};$$

therefore $PQ \equiv 4 \pmod{64}$ and $r \equiv 47 \pmod{64}$,
and hence the only other possible form of H is

$$H_n = 20{,}000n + 14{,}851.$$

We have also the following residues for the respective moduli and the corresponding possible forms of H:—

Mod	7.	9.	11.	13.	17.	19.	23.	29.	31.
N	1.	4.	7.	5.	12.	6.	10.	2.	25.
H±	1,	2,	0,	1,	2,	2,	0,	1,	1,
	3.		1,	2,	4,	5,	4,	3,	2,
			4.	3.	5,	6,	6,	5,	5,
					8.	7,	7,	6,	8,
						8.	8,	8,	9,
							9.	13,	13,
								14,	14.
									15.

Thus, combining $H = 9m \pm 2$ with H_m and H_n, we are enabled to proceed from four starting points by steps of 180,000, and using $7m \pm 1$, 3 from sixteen starting points by steps of 1,260,000. In this manner the following values of m and n were obtained:—

$m = 467$; 1459; 1864; 2179; 4924; 5392; 6074; 10,936; 12,644; 14,455; 15,031; 15,101; 16,102; 17,090.

$n = 1787$; 2291; 2991; 3189; 5214; 7032; 7331; 10,380; 10,794; 17,220: 18,032; 18,806.

It is easy to see that for a square ending we have the following scheme:— $\begin{pmatrix}5\\7\\9\end{pmatrix}$ 4851 and $\begin{pmatrix}3\\5\\7\end{pmatrix}$ 6101. Thus the numbers underlined are excluded, leaving 12 trial values of H none of which are such that $H^2 - N = \square$. Thus N is determined as prime.

Note on Quest. 14254 *by the* Proposer.

To express N in the form $a^2 - b^2$, when $N = 78{,}875{,}943{,}472{,}201$, a factor of $(10^{25}+1) \div (10^5+1)$, since any factor of this number $\equiv 1$, or 51 (mod 200), $(a+b) \equiv (a-b) \equiv 1$, or 51 (mod 200); therefore $b \equiv 0$ (mod 100), also $N \equiv 1 \pmod 3$, $b \equiv 0 \pmod 3 \equiv 0 \pmod{300}$; then, considering the different forms of $b + 300$,

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*(i.) $b-3000\beta = \pm 300$, $N = a^2-(40B+1)(300)^2$,

$a-1,800,000\alpha = 176,101$; 576,101; 1,469,851; 69,851.

*(ii.) $b-3000\beta = \pm 900$, $N = a^2-(40B+9)(300)^2$,

$a-1,800,000\alpha = 536,101$; 936,101; 29,851; 429,851.

*†(iii.) $b-3000\beta = 1500$, $N = a^2-(8B+1)(1500)^2$,

(1) $b-3000\beta = \pm 600$, $N = a^2-(5B+1)(600)^2$,

$a-900,000\alpha = 311,101$; 711,101; 254,851; 654,851.

(2) $b-3000\beta = \pm 1200$, $N = a^2-(5B+4)(600)^2$,

$a-900,000\alpha = 851,101$; 351,101; 794,851; 294,851.

(3) $b-3000\beta = 0$, $N = a^2-B(3000)^2$,

$a-4,500,000\alpha = 131,101$; 4,131,101; 974,851; 474,851.

(1), (2), (3) may, each of them, be extended by separating the case of $b+600$ = an even number from that of $b+600$ = an odd number. (3) can be further extended by considering the different forms of $b+3000$.

These data exclude 8 out of the 12 final trial values found by Mr. BIDDLE, which were as follows:—

77,441,101;	227,769,851;	341,816,101;	350,124,851;
104,294,851;	236,724,851;	344,414,851;	354,579,851;
218,736,101;	240,741,101;	344,949,851;	380,246,101;

because only 4 of their last 5 figures are right; 2 more (227,679,851; 344,949,851) because only 5 out of their last 6 figures are right; also $104,294,851 \equiv 7 \pmod{27}$, whereas $a = \pm 11 \pmod{27}$ or $\pm 25 \pmod{81}$; and is therefore excluded.

[The above N has been independently investigated at great expenditure of time and labour not only by Rev. J. CULLEN and Mr. BIDDLE, as seen above, but by Lt.-Col. ALLAN CUNNINGHAM, R.E., who agrees as to its being a prime. Another form of it is $4,814,017^2+2\cdot5,277,366^2$. See *Nouvelles Annales de Mathématiques*, Tome XV., p. 225; and *Messenger of Mathematics*, Vol. XXV., p. 35.]

Note on Expressing a Number as the Difference of Two Squares.
By C. E. BICKMORE, M.A.

Since $N = \{\frac{1}{2}(N+1)\}^2-\{\frac{1}{2}(N-1)\}^2$ (identically),

the equation $N = a^2-b^2$ may be written

$$a^2 = \{\tfrac{1}{2}(N+1)\}^2-[\{\tfrac{1}{2}(N-1)\}^2-b^2].$$

---

* All these may be extended by noting that either

$a-9,000,000\alpha = 1,256,101$; 4,256,101; 4,349,851; 8,349,851.

$a \equiv \pm 11 \pmod{27}$, $b = 3(3\beta \pm 1)$;

or $a \equiv \pm 25 \pmod{81}$, $b = 9\beta$.

† (iii.) is capable of further extension, on considering the different forms of $b+1500$.

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Whenever it can be shown that b is *always* divisible by q, $\frac{1}{2}(N-1)$ (which is a *particular* value of b) is also divisible by q; therefore

$$a^2 = \left\{\tfrac{1}{2}(N+1)\right\}^2 - Mq^2,$$

where M is an arbitrary integer. *E.g.*, let $q = 300$ (as in the case when $N = 78,875,943,472,201$); therefore

$$a^2 = \left\{\tfrac{1}{2}(N+1)\right\}^2 - M(300)^2, \quad a^2 \equiv \left\{\tfrac{1}{2}(N+1)\right\}^2 \pmod{90,000}.$$

Before taking the square root of each side, the different prime factors of q must be considered separately, because the signs for different moduli may be different, thus

$$a \equiv \pm\tfrac{1}{2}(N+1) \pmod{1250} \equiv \pm\tfrac{1}{2}(N+1) \pmod 8 \equiv \pm\tfrac{1}{2}(N+1) \pmod{18};$$

but the lower sign is excluded for the modulus 1250 in this case, since it is known that $a \equiv +1 \pmod{50}$, therefore

$$a \equiv 1101 \pmod{1250} \equiv \pm 3 \pmod 8 \equiv \pm 7 \pmod{18};$$

combining the first two congruences, $a \equiv 1101$ or $4851 \pmod{5000}$.

But, again, $\frac{1}{2}b$ is either an even or an uneven integer. $\frac{1}{2}(N-1)$ is a value of b of the latter kind; therefore whenever

$$b = 8\beta + 4, \quad b^2 \equiv \left\{\tfrac{1}{2}(N-1)\right\}^2 \equiv 16 \pmod{128},$$

and
$$a^2 \equiv \left\{\tfrac{1}{2}(N+1)\right\}^2 \pmod{128}:$$

but
$$N = 128n + 73; \quad \therefore \quad a \equiv \pm 37 \pmod{64},$$

when $\frac{1}{2}b$ is uneven. If $\frac{1}{2}b$ be even, $a^2 \equiv N \pmod{64}$ suffices; $b^2 \equiv 0 \pmod{64}$; and then $a^2 \equiv 9 \pmod{64}$, giving $a \equiv \pm 3 \pmod{32}$.

This *second* case can be divided into *two* others (as Mr. LAWRENCE shows):—

(1) $\frac{1}{2}b$ uneven, giving $a^2 \equiv N \pmod{512}$, $a \equiv \pm\alpha \pmod{256}$.

(2) $\frac{1}{2}b$ even, $a^2 \equiv N \pmod{256}$, $a \equiv \pm\alpha' \pmod{128}$.

(1) Combining $a \equiv 37 \pmod{64}$ with $a \equiv 1101 \pmod{5000}$, one gets

$$\left.\begin{matrix} 5000a + 1101 \equiv 37 \\ 5000a + 1064 \equiv 0 \end{matrix}\right\} \pmod{64}, \qquad \left.\begin{matrix} 625a + 133 \equiv 0 \\ a + 5 \equiv 0 \end{matrix}\right\} \pmod 8, \quad a \equiv 3;$$

therefore
$$a \equiv 16,101 \pmod{40,000};$$

combining $a \equiv -37 \pmod{64}$ with $a \equiv 4851 \pmod{5000}$,

$$\left.\begin{matrix} 5000a + 4851 \equiv 27 \\ 5000a + 4824 \equiv 0 \end{matrix}\right\} \pmod{64}; \quad \therefore \quad \left.\begin{matrix} 625a + 603 \equiv 0 \\ a + 3 \equiv 0 \end{matrix}\right\} \pmod 8, \quad a \equiv 5;$$

therefore
$$a \equiv 29,851 \pmod{40,000};$$

i.e., $a - 40,000a = 16,101$, or $29,851$, if $\frac{1}{360}b$ be uneven; therefore, since

$$a \equiv \pm 7 \pmod{18}, \quad N = (360,000a + r)^2 - (600\beta + 300)^2,$$

where $r = 216,101$; $176,101$; $29,851$; or $69,851$.

(2) Again, combining $a \equiv 29 \pmod{32}$ with $a \equiv 1101 \pmod{5000}$,

$$\left.\begin{matrix} 5000a + 1101 \equiv 29 \\ 5000a + 1072 \equiv 0 \end{matrix}\right\} \pmod{32}, \qquad \left.\begin{matrix} 625a + 0 \equiv 0 \\ a \equiv 0 \end{matrix}\right\} \pmod 4;$$

therefore
$$a \equiv 1101 \pmod{20,000};$$

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combining $a \equiv 4851 \pmod{5000}$ with $a \equiv 3 \pmod{32}$,

$$\left.\begin{matrix} 5000a + 4851 \equiv 3 \\ 5000a + 4848 \equiv 0 \end{matrix}\right\} \pmod{32}, \qquad \left.\begin{matrix} 625a + 606 \equiv 0 \\ a + 6 \equiv 0 \end{matrix}\right\} \pmod{4}, \quad a \equiv 2;$$

$$a \equiv 14{,}851 \pmod{20{,}000};$$

therefore $a - 20{,}000\alpha = 1101$, or $14{,}851$, if $\frac{1}{800}b$ be even.

Since $a \equiv \pm 7 \pmod{18}$, $N = (180{,}000\alpha + r)^2 - (600\beta)^2$,

where $r = 171{,}101$; $131{,}101$; $114{,}851$; or $74{,}851$.

14196. (Professor Thomas Savage.)—ABC ... WXA is a rectilineal figure inscribed in a circle; the sum of A and any angle other than B or X is equal to two right angles: show that the figure is a quadrilateral.

Solution by H. J. Woodall, A.R.C.S.

Let A and C be supplemental; then AX will meet the circle (called K) in the same point as CD. Supposing A and C not to be supplemental, then, if A and D are so, we shall have AX and DE meeting in a point concyclic with A, D, and the intersection of AB and CD. Now, since, so far, the figure is non-re-entrant, AB and CD will meet outside K on that side, and consequently AX and DE will meet inside K on the other side, and so the figure on this latter side and any continuation thereof will be re-entrant, which seems to be opposed to the terms of the Question. Therefore the figure cannot have more than four sides.

14149. (B. N. Cama, M.A.)—A map consists of $p+1$ groups of land to be coloured, consisting, respectively, of 1, 2, 3, ..., $(p+1)$ countries meeting at a point, the extreme countries in each group not having a common boundary. Show that the total number of ways of painting the map by $k+1$ colours, when no two adjacent countries in any of the groups are to have the same colour, is $(k^{p+1}-1)(k+1)/(k-1)$. Further, show that, if v_r denote the number of ways of painting a group of u countries in this manner, $v_{r+s}/v_s = k^r$.

Solution by the Proposer.

Taking a particular group consisting of n countries, it is clear that, since there are $k+1$ colours at our disposal, the first country can be painted in $k+1$ ways. Having put in a particular colour, the second country can only be painted in k ways, because two adjacent countries are not to have the same colour. In the same way, since the third country is not to have the second colour, but may have the first, there is a choice of k colours left; and therefore it can also be painted in k ways. Similarly, all the other countries can be painted in k ways.

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Therefore $v_n = (k+1) \times k \times k \times \ldots$ repeated $(n-1)$ times $= (k+1) \times k^{n-1}$;
therefore $v_s = (k+1)\, k^{s-1}$; $v_{s+1} = (k+1)\, k^s = k.v_s$.

So $v_{s+2} = k^2.v_s$; therefore $v_{s+r} = k^r.v_s$.

Also the total number of ways of painting the whole map

$$= \sum_{n=1}^{n=p+1} (k+1)\, k^{n-1} = (k+1)\left\{1 + k + k^2 + \ldots + k^p\right\} = \frac{k+1}{k-1}(k^{p+1}-1).$$

5835. (E. B. Seitz.)—Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is $\frac{1}{5}\frac{1}{5}$, and (2) the average distance between them is $\frac{1}{5}\frac{1}{5}r$.

Solution by H. W. Curjel, M.A.

(1) Let $x\ (\not> 3a)$ be the distance of a point P from the centre of a sphere of radius a.

The probability that the distance of P from a random point in the sphere is less than $2a$

$$= \int_{x-a}^{2a} 2\pi r \frac{2xr - x^2 - r^2 + a^2}{2x}\, dr \Big/ \frac{4\pi a^3}{3} = \frac{1}{16a^3 x}(x-3a)^2(x^2+6ax-3a^2);$$

therefore the probability required equals mean value of this probability when P is a random point within the second sphere,

$$= \int_a^{3a} 2\pi x \frac{(4ax-3a^2-x^2)}{4a} \frac{(x-3a)^2(x^2+6ax-3a^2)}{16a^3 x}\, dx \Big/ \frac{4\pi a^3}{3}$$

$$= -\frac{3}{128a^7}\int_{-2a}^{0} y^3(y^3+14y^2a+48ya^2+48a^3)\, dy = \frac{1}{1}\frac{3}{5}.$$

(2) Mean value of distance of a random point in the first sphere from P when P is fixed

$$= \int_{x-a}^{x+a} 2\pi r \left(\frac{2xr - x^2 - r^2 + a^2}{2x}\right) r\, dx \Big/ \frac{4\pi a^3}{3} = \frac{a^2+5x^2}{5x};$$

therefore mean value required equals mean value of this quantity, when P is a random point in the second sphere

$$= \int_a^{3a} 2\pi x \frac{4ax-3a^2-x^2}{4a} \frac{a^2+5x^2}{5x}\, dx \Big/ \frac{4\pi a^3}{3} = \frac{1}{5}\frac{1}{}.$$

Note on Quest. 4834. *By* Professor Sanjána.

The difference in the two solutions of this Question given in Vol. lxx. arises from the fact that on p. 93 it is treated as a problem in Dynamics of a Particle, and on p. 127 as one in Rigid Dynamics. The first solution is not inaccurate in itself.

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14264. (R. TUCKER, M.A.)—P is a point on a parabola, focus S and vertex A. With P as centre and radius PS, a circle is described. The tangent to the parabola cuts this circle in L, M, and AL, AM cut the parabola in Q, R. Find the equations to AL, AM, and to the chord QR.

Solution by H. A. WEBB *and* G. W. PRESTON, B.A.

The parabola is $y^2 = 4ax$, and P is $(a/m^2, 2a/m)$.

LM is $y = mx + a/m$ ..... (1),

and the circle LMS is

$$(x - a/m^2)^2 + (y - 2a/m)^2 = (a + a/m^2)^2$$

or $$x^2 + y^2 - 2a(x/m^2 + 2y/m) + a^2(2/m^2 - 1) = 0 \quad \ldots\ldots (2).$$

The lines joining the origin to the common points of (1) and (2) are found by making (2) homogeneous by means of (1). The result is

$$x^2 + y^2 - 2(x/m^2 + 2y/m)(my - m^2x) + (2/m^2 - 1)(y - mx)^2 m^2 = 0,$$

which reduces to

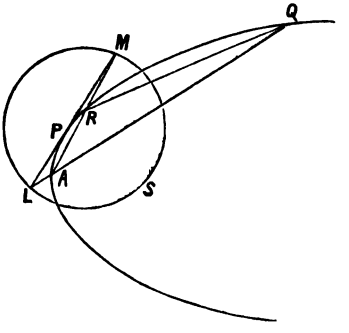
$$x^2(m^2 - 3) - 2xy(m - 1/m) + y^2 = 0 \quad \ldots\ldots (3).$$

This is the equation of AL, AM. (3) may be written in the form

$$y^2 = 4ax(1/4a)\{2y(m - 1/m) - x(m^2 - 3)\},$$

which is the equation of the lines joining the origin to the common points of $y^2 = 4ax$ and $x(m^2 - 3) - 2y(m - 1/m) + 4a = 0$ ..... (4).

(4) is therefore the equation of QR.



14332. (Professor M. W. CROFTON, F.R.S.)—(1) If $4n-1$ is a prime, $1^2 2^2 3^2 \ldots (2n-1)^2 - 1$ is a multiple of $4n-1$. (2) If $4n+1$ is a prime, $1^2 2^2 3^2 \ldots (2n)^2 \cdot 1$ is a multiple of $4n+1$. It hence follows that $4n+1$ is the sum of two squares.

Solution by G. H. HARDY, B.A.; *and* Lt.-Col. ALLAN CUNNINGHAM, R.E.

If $2p-1$ be prime, then, by WILSON'S theorem,

$$2p-2\,! + 1 \equiv 0 \pmod{2p-1}.$$

Now $$2p-2-s \equiv -(s+1), \quad (s = 0, 1, \ldots p-2),$$

therefore $p.p+1 \ldots 2p-2 \equiv (-)^{p-1} p-1\,!$, $2p-2\,! \equiv (-)^{p-1}(p-1\,!)^2$.

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So, if $p = 2n$, $(2n-1!)^2 - 1 \equiv 0 \pmod{4n-1}$;

if $p = 2n+1$, $(2n!)^2 + 1 \equiv 0 \pmod{4n+1}$.

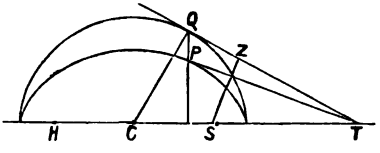
Thus a prime of the form $4n+1$ is a divisor of a number which is the sum of two squares, and is therefore, by a known theorem, itself the sum of two squares.

14354. (H. W. Curjel, M.A.)—If the ordinate of a point P on an ellipse cuts the auxiliary circle in Q, the distance of the focus S from the tangent at Q = SP.

Solution by G. Birtwistle, B.A., B.Sc.; J. H. Taylor, M.A.; F. H. Peachell, B.A.; *and others.*

Let QT, PT be tangents to circle and ellipse at Q and P respectively. Let H be the other focus, and SZ the perpendicular from S on QT.

Then, if C is the centre of the ellipse,



$$\frac{ST}{CT} = \frac{SZ}{CQ} = \frac{SZ}{AC}.$$

Since PT bisects the exterior angle at P of the triangle SPH,

$$\frac{SP}{HP} = \frac{ST}{HT}; \quad \therefore \textit{componendo}, \quad \frac{SP}{AC} = \frac{ST}{CT}; \quad \therefore \quad SZ = SP.$$

14341. (J. A. Third, M.A.)—K is a conic circumscribed to the triangle of reference and passing through the centroid; K′ is another conic passing through the mid-points of the sides and touching K at the centroid; P is the centre of a conic which touches the sides at points which connect with the opposite vertices through a point on K; Q is the point of concurrence of lines drawn from the vertices to meet the opposite sides at points where these sides are touched by a conic whose centre lies on K′. Show that PQ touches K and K′ at the centroid.

Solution by Rev. J. Cullen.

We have $K \equiv \Sigma l\beta\gamma = 0$ where $\Sigma al = 0$, $K' \equiv \Sigma l' (S - a\alpha)^{-1} = 0$ where $2S = \Sigma a\alpha$ and $\Sigma l' = 0$; also the tangent to K at the centroid is $\Sigma a^2 l\alpha = 0$. Hence K′ is also of the form $K - \Sigma a^2 l\alpha \Sigma p\alpha = 0$, since they touch. Determining p, &c. by the condition that K′ passes through the mid-points of the sides, and comparing with the previous form, we find $l' = al$, &c. Now let $\alpha/\alpha' = \ldots = \ldots$ be the point on K of concurrence of the connectors; then the equation of the in-conic is $\Sigma (\alpha/\alpha')^{\frac{1}{2}} = 0$ with $\Sigma l\beta'\gamma' = 0$. Hence

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P is the point $a/a'(c\gamma' + b\beta') = \ldots = \ldots$, so that $a(S - aa) = k\beta'\gamma'$, &c.; hence, substituting in $\Sigma l\beta'\gamma' = 0$, we see that P lies on the tangent to K at the centroid. Similarly, from $K' \equiv \Sigma al(S - aa)^{-1} = 0$, we see that Q also lies on this tangent. Therefore &c.

14324. (W. S. Cooney.)—P, Q, R are the centres of the insquares of the triangle ABC, to the sides a, b, c respectively. (1) AQ and AR are isogonal lines to b and c, &c. (2) If triangle be right-angled, given P, Q, and centre of inscribed circle, construct triangle. (3) In (2), if P and Q alone be given, area of minimum right-angled triangle equals $36(PQ)^2$.

Solution by the Proposer.

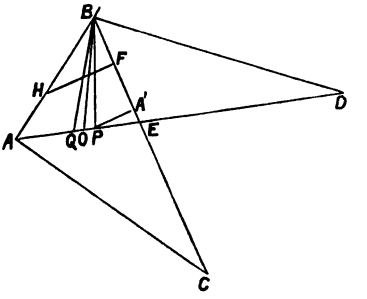
(1) If ABC be any triangle, and P and Q the centres of the insquares to BC and AB, then

$$\cot PBC = \frac{BA'}{PA'} = \frac{BF + FA'}{PA'} = \frac{S\cot B + \frac{1}{2}S}{\frac{1}{2}S}$$

(S = side of square HF)

$= 1 + 2\cot B = \cot QBA$.

Therefore &c.



(2) If ABC be right-angled at A, then P and Q lie on AE, the bisector of $\angle A$, Q being the middle point of AE. If O be the incentre, then, from (1), BO bisects $\angle PBQ$ and $\angle ABC$; and if D be taken so that $QO : OP = QD : DP$, and A and E be taken so that $QA^2 = QE^2 = QO \cdot QD$, then A and E, Q and P are harmonic conjugates of O and D, and the circle on OD as diameter is locus of the points B and C subtending equal angles at AQ and EP: therefore, if this circle be drawn and AB be drawn making $\angle BAE = 45°$, triangle can be constructed. Q and P then are centres of insquares, since $AQ = \frac{1}{3}AE$ and $\angle ABQ = EBP$.

(3) For any given value of AE the triangle is a minimum when isosceles, but, when ABC is isosceles, $AE = 6PQ$; for, from (1), $\cot PBE$ then $= 1 + 2\cot 45° = 3$; therefore $PE = \frac{1}{3}BE = \frac{1}{3}AE$; therefore area of minimum right-angled triangle equals $36(PQ)^2$.

14241. (Rev. T. Roach, M.A. (Suggested by Quest. 13173.)—Six circles are described, each passing through the same excentre of a triangle, and touching one side of the triangle at one of its extremities. (1) Show that the centres of the six circles lie on the radii I_aD, ... on three circles

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concentric with the excircle. (2) Show that the product of the radii of three alternate circles $= 2R^2r_a$.

Solution by the PROPOSER *and* LIONEL E. REAY, B.A.

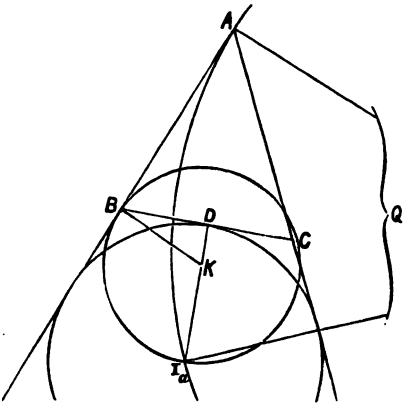
Let K be the centre; then, as in 14208, K lies on I_aD, if circle touches AB in B,

$$I_aK = \frac{I_aB}{2\cos\frac{1}{2}B} = \frac{r_a}{2\cos^2\frac{1}{2}B}.$$

If the circle touches BC in B, it has centre on I_aF and

$$\text{radius} = \frac{r_a}{2\cos^2\frac{1}{2}B}.$$

If it touches AC or BC in C, centre lies on I_aD or I_aF and radius $= \frac{r_a}{2\cos^2\frac{1}{2}C}$.



Again, for a circle touching AB in A, if Q be the centre, $AQI_a = A$.

Therefore $$AI_aQ = 90 - \tfrac{1}{2}A = AI_aE.$$

Therefore centre lies on I_aE and

$$I_aQ = \frac{AI_a}{2\sin\frac{1}{2}A} = \frac{r_a}{2\sin^2\frac{1}{2}A}.$$

For circle touching AC in A, centre lies on I_aF and

$$\text{radius} = \frac{r_a}{2\sin^2\frac{1}{2}A}.$$

Therefore product of alternate three radii

$$= \frac{r_a^3}{8\sin^2\frac{1}{2}A\cos^2\frac{1}{2}B\cos^2\frac{1}{2}C} = 2R^2r_a.$$

Similarly for circles through I_b and I_c.

14306. (Professor G. B. MATHEWS, F.R.S.)—Prove that, if p, q be odd primes such that p lies between q and $3q$, and if p'/q' be the penultimate convergent in the expansion of p/q as an ordinary continued fraction, then there is a multiple of pq between $(p+q)pq'$ and $(p+q)p'q$. Show, also, that $p+q$ is the least even number $2n$ such that a multiple of pq lies between $2npq'$ and $2np'q$.

Show, further, that, if $2n$ be an assigned even integer, and if odd primes p, q can be found such that $p > \frac{1}{2}n$, $q > \frac{1}{2}n$, and a multiple of pq lie between $2npq'$ and $2np'q$ when $pq' - p'q = \pm 1$, then $2n = p+q$.

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Solution by H. W. CURJEL, M.A.

$$p'q - pq' = \pm 1 = e \text{ (say)};$$

therefore $\{(p+q)\,pq'\}/pq = pq'/q + q' = p' + q' - e/q$

and $\{(p+q)\,p'q\}/pq = p'q/p + p' = p' + q' + e/p.$

Hence a multiple of pq, namely, $pq\,(p'+q')$, lies between $(p+q)\,pq'$ and $(p+q)\,p'q$. Let a multiple of pq lie between $2np'q$ and $2npq'$, where $2n$ is not a multiple of $(p+q)$.

Then $2np'q/pq = \{2n\,(p'+q')\}/(p+q) + 2ne/\{pq\,(p+q)\}$

and $2npq'/pq = \{2n\,(p'+q')\}/(p+q) - 2ne/\{pq\,(p+q)\}.$

Hence $2np'q/pq$ and $2npq'/pq = x + (ypq \pm 2n)/\{(p+q)\,pq\}$,

where x is an integer and $y < p+q$; therefore $ypr < 2n$ or $2n + ypq > (p+q)\,pq$, *i.e.*, $2n > pq > 2\,(p+q)$ (except in the case $q = 3$, $p = 5$), for $(p-2)(q-2) > 4$; therefore $p+q$ is the least even number satisfying the given conditions, and the next is $2\,(p+q)$.

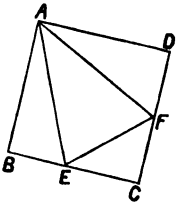
Hence also the last part of the question follows immediately, since $2\,(p+q) > 2n$.

The case $q = 3$, $p = 5$ can easily be verified independently.

14113. (D. BIDDLE.)—From one angle of a parallelogram draw a straight line to each of the opposite sides, such that, with the join of their distal extremities, three equal triangles may be cut off; and prove that the area of the remaining (central) triangle is to one of the former as $\sqrt{5} : 1$.

Solution by V. R. THYAGARAGAIYAR *and* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(1) The problem is easily solved for a square, thus:—



Let AE, AF be the straight lines required, and let $BE/EC = 1/\lambda$.

Then the conditions give

$$\lambda^2 = \lambda + 1.$$

Therefore $\lambda = \frac{1}{2}(1 + \sqrt{5})$,

the negative sign being inadmissible. The area of the remaining triangle: the area of one of the equal triangles

$$= \frac{(\lambda+1)^2 - \frac{3}{2}(\lambda+1)}{\lambda^2/2} = \sqrt{5} : 1.$$

(2) Project the square orthogonally on any plane not parallel to the plane of the square, and the results for a parallelogram are obtained.

[By way of generalizing, it may be observed that, if lines be drawn from an angle of a parallelogram as above, cutting off two equal triangles, and the join of their distal extremities cut off a triangle m times as great, the relative area of the central triangle is $(m^2 + 4m)^{\frac{1}{2}}$.]

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14330. (R. Chartres.)—BC is the fixed base of the triangle ABC, G the centroid, and K the symmedian point. Show that the maximum value of the triangle AGK : its mean value :: π : 2 if A describe a semicircle.

Solution by Rev. J. Cullen.

$AG = (2K - 3a^2)^{\frac{1}{2}}$, $AK = \{bc(2K - 3a^2)^{\frac{1}{2}}\}/K$, where $K = a^2 + b^2 + c^2$.

Also $\sin GAK = 2\Delta(c^2 - b^2)/bc(2K - 3a^2)$.

Therefore the area of the triangle $AGK = \Delta(c^2 - b^2)/3K$, if $\angle ABC = \theta$.

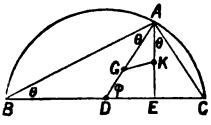
Then $b = a\cos\theta$, $c = a\sin\theta$.

Therefore area of AGK varies as $\sin 4\theta$, the maximum value of which is 1.

$$\text{Mean value} = -\int_{\frac{1}{4}\pi}^{\pi} \sin 4\theta \, d\theta \Big/ \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} d\theta = 2/\pi.$$

Therefore, &c.

[The Proposer solves the Question thus:—"Here the symmedian is the altitude, and K is its mid-point; thus area of AGK = $\frac{1}{3}$ of $\frac{1}{4}$ of ADE, whose maximum value evidently is $\frac{1}{16}a^2$, and mean value $a^2/24\pi$."]



14319. (Rev. T. Mitchesson, B.A.)—Solve

$$x^3 + \tfrac{1}{2}x^2(\alpha^{-1}\beta\gamma + \alpha\beta^{-1}\gamma + \alpha\beta\gamma^{-1}) = \tfrac{1}{2}\alpha\beta\gamma.$$

Solution by Constance I. Marks, B.A.

Let ξ_1, ξ_2, ξ_3 be the roots. Then

$$\xi_1 + \xi_2 + \xi_3 = -\tfrac{1}{2}\alpha\beta\gamma(1/\alpha^2 + 1/\beta^2 + 1/\gamma^2) \quad \text{.....(i.),}$$

$$\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 = 0, \quad (\xi_1\xi_2\xi_3) = \tfrac{1}{2}\alpha\beta\gamma \quad \text{.....(ii., iii.).}$$

From (ii.) $\xi_2 + \xi_3 = -\xi_2\xi_3/\xi_1 = -\alpha\beta\gamma/2\xi_1^2$.

Hence (i.) may be written

$$\xi_1 - \alpha\beta\gamma/2\xi_1^2 + \tfrac{1}{2}\alpha\beta\gamma(1/\alpha^2 + 1/\beta^2 + 1/\gamma^2) = 0,$$

$$2/\alpha\beta\gamma + (1/\alpha^2 + 1/\beta^2 + 1/\gamma^2)\,1/\xi_1 - 1/\xi_1^3 = 0,$$

$$1/\xi_1^3 - (1/\alpha^2 + 1/\beta^2 + 1/\gamma^2)\,1/\xi_1 - 2/\alpha\beta\gamma = 0.$$

Put $1/\xi_1 = \eta + \zeta$ and solve by Cardan's method. This gives $1/\xi_1$, and therefore ξ_1 *any* root of the given equation.

5893. (W. R. Roberts, M.A.)—A system of conics is described having double contact with the polar conic of the line joining the three inflexious of a cubic, the chord of contact being the same line; any conic of the system meets the cubic in six points, the tangents at which all touch another conic of the same system.

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Solution by Professor G. B. MATHEWS, F.R.S.

Take the cubic in the form

$$(x+y+z)^3+6axyz=0 \quad \ldots\ldots (1),$$

so that

$$x+y+z=0 \quad \ldots\ldots (2)$$

is the line joining the inflexions. The polar conic of ξ, η, ζ is

$$(\xi+\eta+\zeta)(x+y+z)^2+2a(\xi yz+\eta zx+\zeta xy)=0 \quad \ldots\ldots (3),$$

and this touches (2) if $\xi^2+\eta^2+\zeta^2-2\eta\zeta-2\zeta\xi-2\xi\eta=0$.

Hence the polar conic of $x+y+z=0$ is

$$P \equiv x^2+y^2+z^2-2yz-2zx-2xy=0.$$

The system of conics having double contact with P, where $x+y+z=0$, may be written

$$x^2+y^2+z^2+2\lambda(yz+zx+xy)=0 \quad \ldots\ldots (4).$$

The tangent to the cubic at (ξ, η, ζ) is

$$(\xi+\eta+\zeta)^2(x+y+z)+2a(\eta\zeta x+\zeta\xi y+\xi\eta z)=0$$

or

$$ux+vy+wz=0$$

if

$$u=(\xi+\eta+\zeta)^2+2a\eta\zeta=0, \text{ \&c.} \quad \ldots\ldots (5).$$

This touches (4) if

$$(1-\lambda^2)(u^2+v^2+w^2)+2\lambda(\lambda-1)(vw+wu+uv)=0$$

or

$$(\lambda+1)(u^2+v^2+w^2)+2\lambda(vw+wu+uv)=0 \quad \ldots\ldots (6).$$

Now, if (ξ, η, ζ) is an intersection of the cubic with

$$x^2+y^2+z^2+2\mu(yz+zx+xy)=0,$$

and we put

$$\xi+\eta+\zeta=\sigma,$$

then

$$\Sigma\eta\zeta=-\frac{1}{2(\mu+1)}\sigma^2, \quad \xi\eta\zeta=-\frac{1}{6a}\sigma^3.$$

Hence

$$u=\sigma^2+2a\eta\zeta, \text{ \&c.},$$

$$\Sigma u^2=3\sigma^4+4a\sigma^2\Sigma\eta\zeta+4a^2\Sigma\eta^2\zeta^2=3\sigma^4-\frac{2a}{\mu+1}\sigma^4+4a^2\left\{\frac{\sigma^4}{4(\mu+1)^2}+\frac{1}{3a}\sigma^4\right\}$$

$$=\left(3+\frac{4a}{3}-\frac{2a}{\mu+1}+\frac{a^2}{(\mu+1)^2}\right)\sigma^4;$$

$$vw+\ldots+\ldots=3\sigma^4+4a\sigma^2\Sigma\eta\zeta+4a^2\sigma.\xi\eta\zeta=\left(3-\frac{2a}{\mu+1}-\frac{2a}{3}\right)\sigma^4.$$

Substituting in (6), the result is an identity if

$$(\lambda+1)\left\{3+\frac{4a}{3}-\frac{2a}{\mu+1}+\frac{a^2}{(\mu+1)^2}\right\}+2\lambda\left(3-\frac{2a}{3}-\frac{2a}{\mu+1}\right)=0;$$

i.e.,

$$\lambda\left(3-\frac{a}{\mu+1}\right)^2+\frac{3a^2-6(\mu+1)a+(4a+9)(\mu+1)^2}{(\mu+1)^2}=0;$$

i.e.,

$$(\lambda+1)\{3(\mu+1)-a\}^2+2a\{a+2(\mu+1)^2\}=0.$$

Changing the notation by putting λ, μ instead of $\lambda+1, \mu+1$, the associated conics are

$$x^2+y^2+z^2+2(\mu-1)(yz+zx+xy)=0,$$

$$x^2+y^2+z^2+2(\lambda-1)(yz+zx+xy)=0,$$

with

$$\lambda=-\frac{2a(a+2\mu^2)}{(3\mu-a)^2}.$$

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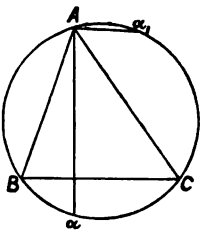
14228. (Professor A. Droz-Farny.)—Un triangle ABC est inscrit dans une conique. Les deux bissectrices de l'angle A rencontrent pour la seconde fois en α et α' la courbe. Démontrer que la droite $\alpha\alpha'$ et ses analogues $\beta\beta'$, $\gamma\gamma'$ se coupent en un même point.

Solutions (1) *by* J. O. Watts *and* Professor Sanjána, M.A.;
(2) *by* H. W. Curjel, M.A.

(1) $A\alpha$ is $\beta = \gamma$, $A\alpha_1$ is $\beta = -\gamma$.



Conic is $l/\alpha + m/\beta + n/\gamma = 0$;

therefore $l/\alpha = -(m+n)/\beta = -(m+n)/\gamma$ at a,
$l/\alpha = -(m-n)/\beta = (m-n)/\gamma$ is a_1;

therefore aa_1 has equation

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & -(m+n) & -(m+n) \\ l & -(m-n) & (m-n) \end{vmatrix} = 0;$$

or $$\alpha(m^2-n^2) + \beta ml + \gamma nl = 0.$$

$\beta\beta_1$ is $$-aml + \beta(n^2-l^2) + \gamma mn = 0.$$

$\gamma\gamma_1$ is $$anl - \beta nm + \gamma(l^2-m^2) = 0.$$

The condition that these meet at a point is

$$\begin{vmatrix} m^2-n^2 & ml & -nl \\ -ml & n^2-l^2 & mn \\ -nl & -nm & l^2-m^2 \end{vmatrix} = 0, \quad \text{which is true.}$$

(2) Let $lyz + mzx + nxy$ be any conic circumscribing the triangle of reference ABC, and let xyz be any point P, and let AP, BP, CP meet the conic again in α, β, γ, and let α' be a point on the conic such that A(BC, $a\alpha'$) is harmonic, and let β' and γ' be similarly determined. a and α' are given by

$$x/\{-ly_1z_1\} = y/\{y_1(ny_1+mz_1)\} = z/\{z_1(ny_1+mz_1)\},$$
$$x/\{ly_1z_1\} = y/\{y_1(ny_1-mz_1)\} = z/\{z_1(mz_1-ny_1)\};$$

therefore equation to $a\alpha'$ is

$$x(m^2/y_1^2) - n^2/z_1^2 + y(lm/x_1y_1) - z(ln/x_1z_1) = 0,$$

or $$x(\mu^2-\nu^2) + y\lambda\mu - z\lambda\nu = 0,$$

where λ, μ, $\nu = l/x_1$, m/y_1, n/z_1, and equations to $\beta\beta_1$, $\gamma\gamma_1$ are

$$-x\mu\lambda + y(\nu^2-\lambda^2) + z\mu\nu = 0, \quad x\nu\lambda - y\nu\mu + z(\lambda^2-\mu^2) = 0,$$

and $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ meet in a point, since

$$\begin{vmatrix} \mu^2-\nu^2, & \lambda\mu, & -\lambda\nu \\ -\lambda\mu, & \nu^2-\lambda^2, & \mu\nu \\ \lambda\nu, & -\mu\nu, & \lambda^2-\mu^2 \end{vmatrix} = 0.$$

If P be the in-centre of ABC, this becomes the theorem of the Question, which may be written: If a conic through the feet of the perpendiculars of a triangle cut the sides in α, β, γ, and the corresponding perpendiculars in α', β' γ', then $a\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent.

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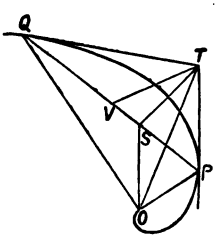
14207. (B. N. Cama, M.A. In continuation of Quest. 14177.)—In the case when $\beta = \frac{1}{2}\pi$, it is evident that a parabola can be described touching the spiral at the points of contact, and having the chord of contact for a focal chord. Prove that (1) the latus rectum of the parabola is $PQ \sin^2\phi$, PQ being the chord of contact and ϕ the inclination of PQ to the axis of the curve; (2) the angle ϕ is constant for all pairs of orthogonal tangents to the spiral, it being equal in fact to $2(\psi - \alpha)$, where ψ is as defined in Quest. 14177; (3) the locus of the focus of the parabola for varying positions of PQ is a copolar spiral.

Solution by the Proposer.



(i.) In the case when $\beta = \frac{1}{2}\pi$, a parabola can be described touching the spiral at P, Q, and having PQ for a focal chord. The focus S of this parabola is found to be the foot of the perpendicular TS from T on PQ, and, V being the mid-point of PQ, TV is the direction of the axis; therefore

$$\angle TVQ = \phi.$$

The semilatus rectum of the curve is by conic sections

$$= \frac{2SP \cdot SQ}{PQ} = \frac{2ST^2}{PQ} = \frac{2t_1^2 t_2^2}{PQ^3},$$

since $$ST \cdot PQ = TP \cdot TQ = t_1 \cdot t_2.$$

Also $$t_1^2 = 2PV^2 + 2PV^2\cos\phi, \quad t_2^2 = 2PV^2 - 2PV^2\cos\phi;$$

therefore $$t_1^2 t_2^2 = 4PV^4 \sin^2\phi;$$

therefore semilatus rectum $= \dfrac{8PV^4 \sin^2\phi}{PQ^3} = \frac{1}{2}PQ\sin^2\phi.$

(ii.) From the figure

$$\angle\phi = \angle TVQ = 2\angle TOQ = 2(\theta_1 - \Theta) = 2(\psi - \alpha),$$

from the first part.

From the same figure

(iii.) $$OS^2 = \rho^2 + ST^2 - 2\rho \cdot ST\cos STO$$
$$= \rho^2 + \tfrac{1}{4}PQ^2\sin^2\phi + PQ^2\sin\alpha\sin\phi\cos(2\psi - \alpha),$$

since $$ST = \tfrac{1}{2}PQ\sin\phi, \quad \rho = PQ\sin\alpha,$$

and $$\angle STO = \tfrac{1}{2}\pi - (\angle SQO + TOQ)$$
$$= \tfrac{1}{2}\pi - \{\alpha - (\tfrac{1}{2}\pi - \tfrac{1}{2}\phi) + \psi - \alpha\} = \pi - \tfrac{1}{2}\phi - \psi$$
$$= \pi - \psi + \alpha - \psi = \pi - 2\psi + \alpha = \text{constant}.$$

Thus $OS^2 = PQ^2\sin^2\alpha + \frac{1}{4}PQ^2\sin^2\phi + PQ^2\sin\alpha\sin\phi\cos(2\psi - \alpha) \propto PQ^2 \propto \rho^2.$

Also $$\cos\angle OST = \frac{\kappa_1\rho^2 + \kappa_2\rho^2 - \rho^2}{2\sqrt{(\kappa_1\kappa_2)}\,\rho^2} \text{ (where } \kappa_1, \kappa_2 \text{ are constants)}$$
$$= \text{constant};$$

therefore $\angle OST$ is constant;

therefore $$\angle SOT = \pi - \angle OST - OTS = \text{constant};$$

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therefore, if Θ be the vector angle of OS, $\Theta = \theta + \lambda$, θ being the vector angle of T and λ a constant; therefore

$$\text{OS} \propto \rho \propto e^{\theta \cot \alpha + \phi \cot \alpha} \propto e^{(\theta - \lambda + \phi) \cot \alpha}.$$

Thus the locus of S is a copolar spiral.

14287. (Professor Thomas Savage.)—Find the sums of the angles made with the axis of x by the lines joining the origin to the points of intersection of (1) $x+y=1$ and $x^3-x^2y-xy^2+y^3=0$; (2) of $x+y=1$ and $x^3-x^2y+xy^2+y^3=0$.

Solution by H. J. Woodall, A.R.C.S.

(1) Gives $x+y=1$ and $(x+y)(x-y)^2=0$; the points of intersection are $(\infty, -\infty)$ and $(\frac{1}{2}, \frac{1}{2})$ bis; the angles made with the axis x by lines joining these points to the origin are respectively $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$ bis, and the required sum is $+\frac{1}{4}\pi$.

(2) Substitute in the cubic $x=1-y$, and we get $2y^3-6y^2+4y-1=0$, which has three roots, y_1, y_2, y_3. If ϕ_1, ϕ_2, ϕ_3 be the three angles, then $\tan\phi = y/(1-y)$.

$$\tan(\phi_1+\phi_2+\phi_3)$$
$$= \frac{\tan\phi_1+\tan\phi_2+\tan\phi_3-\tan\phi_1\tan\phi_2\tan\phi_3}{1-\tan\phi_2\tan\phi_3-\tan\phi_3\tan\phi_1-\tan\phi_1\tan\phi_2}$$
$$= \frac{y_1(1-y_2)(1-y_3)+y_2(1-y_3)(1-y_1)+y_3(1-y_1)(1-y_2)-y_1y_2y_3}{(1-y_1)(1-y_2)(1-y_3)-y_2y_3(1-y_1)-y_3y_1(1-y_2)-y_1y_2(1-y_3)}$$
$$= \frac{(y_1+y_2+y_3)-2(y_2y_3+y_3y_1+y_1y_2)+2y_1y_2y_3}{1-(y_1+y_2+y_3)+2y_1y_2y_3} = \frac{3-4+1}{1-3+1} = 0;$$

therefore $\phi_1+\phi_2+\phi_3 = 0$ or π.

[*Otherwise:* In case (1) the three lines $x^3-x^2y-xy^2+y^3=0$ pass through the origin. Therefore these lines are the lines joining the origin to the points of intersection of themselves and $x+y=1$.

If the angles made with the axis of x be ϕ_1, ϕ_2, ϕ_3, then

$$\Sigma\tan\phi_1 = 1, \quad \Sigma\tan\phi_1\tan\phi_2 = -1, \quad \tan\phi_1\tan\phi_2\tan\phi_3 = -1;$$

therefore $\phi_1+\phi_2+\phi_3 = \tan^{-1}\left(\frac{\Sigma\tan\phi_1-\tan\phi_1\tan\phi_2\tan\phi_3}{1-\Sigma\tan\phi_1\tan\phi_2}\right) = \tan^{-1}1.$

Similar remarks applying in case (2), we have $\phi_1+\phi_2+\phi_3 = \tan^{-1}0$.]

14314. (R. Tucker, M.A.)—AC, BD are two parallel straight lines which are cut at right angles by OAB, and obliquely by OCD ($\angle$ COA $= \theta$). CP, DQ are parallel to OAB, and DP, CQ perpendicular to OCD. Prove loci of P, Q to be parabolas. Tangents at extremities of their latera recta are perpendicular (or parallel), and the points of intersection of these tangents lie on a hyperbola when BD varies in position.

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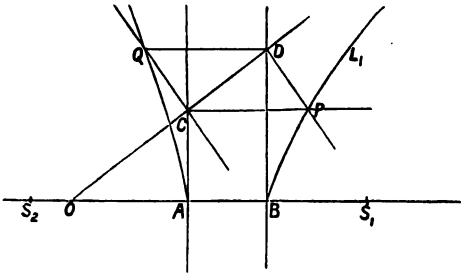
Solution by the PROPOSER.

OAB axis of x, OA $= a$, AB $= b$, CP $(y - a\tan\theta = 0)$ cuts DP $[x + y\tan\theta = (a+b)\sec^2\theta]$ in P,

locus $by^2 = a^2(x-a-b)$ .....(i.),



CQ $(x + y\tan\theta = a\sec^2\theta)$ cuts DQ $[y - (a+b)\tan\theta = 0]$ in Q,

locus $by^2 = -(a+b)^2(x-a)$ .....(ii.),

S_1, S_2 foci of (i.), (ii.) and L_1, L_2 ends of latera recta

$OS_1 = (a + 2b)^2/4b$; $\therefore$ tangent at L_1, $y = x + (a^2 - 4ab - b^2)/4b$...(iii.);

$OS_2 = -(a-b)^2/4b$; $\therefore$ tangent at L_2, $y + x = (a^2 + 6ab + b^2)/4b$ ...(iv.).

If (X, Y) is point of intersection of (iii.), (iv),

$$8b\mathrm{Y} = 2a^2 + 2ab - 3b^2, \quad 8b\mathrm{X} = 5b(2a+b);$$

therefore $5b = 8\mathrm{X} - 10a$;

then locus is $96\mathrm{X}^2 + 160\mathrm{XY} - 200a\mathrm{Y} - 280a\mathrm{X} + 175a^2 = 0$.

6085. (A. J. C. ALLEN, B.A.)—Through the orthocentre K of a triangle ABC a straight line DEF is drawn meeting the sides BC, CA, AB in D, E, F; and a point P is taken on the line so that

$$\frac{\mathrm{PE}\cdot\mathrm{PF}}{\mathrm{KE}\cdot\mathrm{KF}}\sin 2\mathrm{A} + \frac{\mathrm{PF}\cdot\mathrm{PD}}{\mathrm{KF}\cdot\mathrm{KD}}\sin 2\mathrm{B} + \frac{\mathrm{PD}\cdot\mathrm{PE}}{\mathrm{KD}\cdot\mathrm{KE}}\sin 2\mathrm{C} = \text{a constant}.$$

Show that the locus of P is a circle, and find its centre and radius.

Solutions by J. GILBART SMYLY, M.A.

The following analysis will show that this Question is a particular case of a more general problem which may be thus stated:—

Through any point O a straight line DEF is drawn meeting the sides BC, CA, AB of a triangle in D, E, F; and a point P is taken on the line so that

$$\frac{\mathrm{PE}\cdot\mathrm{PF}}{\mathrm{OE}\cdot\mathrm{OF}}\mathrm{L} + \frac{\mathrm{PF}\cdot\mathrm{PD}}{\mathrm{OF}\cdot\mathrm{OD}}\mathrm{M} + \frac{\mathrm{PD}\cdot\mathrm{PE}}{\mathrm{OD}\cdot\mathrm{OE}}\mathrm{N} = \text{a constant} = \mathrm{K}.$$

Determine the conditions that the locus of P may be a circle, and find its centre and radius.

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Take O as origin of coordinates, and let the equations of the sides of the triangle be

$$x\cos\alpha + y\sin\alpha = p_1, \quad x\cos\beta + y\sin\beta = p_2, \quad x\cos\gamma + y\sin\gamma = p_3.$$

The equation of any line through O is $x = \lambda y$. The coordinates of the points P, D, E, F may be written in the form $\lambda p, p$; $\lambda d, d$; $\lambda e, e$; $\lambda f, f$, where $d = \dfrac{p_1}{\lambda\cos\alpha + \sin\alpha}$, &c.

Hence
$$\frac{\text{PE}\,.\,\text{PF}}{\text{OE}\,.\,\text{OF}} = \frac{(p-e)\,(p-f)}{ef} = \frac{\{p\,(\lambda\cos\beta+\sin\beta)-p_2\}\,\{p\,(\lambda\cos\gamma+\sin\gamma)-p_3\}}{p_2p_3}.$$

Substituting these quantities and writing x, y for λp and p respectively, we obtain the equation of the locus of P,

$$\Sigma\frac{(x\cos\beta+y\sin\beta-p_2)\,(x\cos\gamma+y\sin\gamma-p_3)}{p_2p_3}\,\text{L} = \text{K}.$$

The conditions for a circle are

$$\Sigma\frac{\cos(\beta+\gamma)}{p_2p_3}\,\text{L} = 0 \quad \text{and} \quad \Sigma\frac{\sin(\beta+\gamma)}{p_2p_3}\,\text{L} = 0.$$

These are equivalent to $\dfrac{p_1\text{L}}{\sin(\beta-\gamma)} = \dfrac{p_2\text{M}}{\sin(\gamma-\alpha)} = \dfrac{p_3\text{N}}{\sin(\alpha-\beta)} = \dfrac{1}{\theta}$.

If these conditions be fulfilled, the locus is a circle, and its equation is

$$\Sigma\,(x\cos\beta + y\sin\beta - p_2)\,(x\cos\gamma + y\sin\gamma - p_3)\sin(\beta-\gamma) = \theta p_1p_2p_3\text{K}.$$

It is easily seen that the equation of the circumcircle of the triangle is

$$\Sigma\,(x\cos\beta + y\sin\beta - p_2)\,(x\cos\gamma + y\sin\gamma - p_3)\sin(\beta-\gamma) = 0.$$

Hence for all positions of O the centre of the circle coincides with the centre of the circumcircle, and its radius is given by

$$\rho^2 = \text{R}^2 + \theta p_1p_2p_3\text{K}.$$

Let us apply the results when O has certain given positions.

(1) Let O be the centre of the inscribed circle;

$$p_1 = p_2 = p_3 = r, \qquad \frac{1}{\theta} = \frac{p_1\text{L}}{\sin(\beta-\gamma)} = \frac{r\text{L}}{\sin\text{A}};$$

hence, taking $\text{L} = \sin\text{A}$, $\text{M} = \sin\text{B}$, $\text{N} = \sin\text{C}$, $\theta = 1/r$, the locus of P is a circle if $\dfrac{\text{PE}\,.\,\text{PF}}{\text{OE}\,.\,\text{OF}}\sin\text{A} + \dfrac{\text{PF}\,.\,\text{PD}}{\text{OF}\,.\,\text{OD}}\sin\text{B} + \dfrac{\text{PD}\,.\,\text{PE}}{\text{OD}\,.\,\text{OE}}\sin\text{C} = \text{K}$, and its radius is given by $\rho^2 = \text{R}^2 + r^2\text{K}$.

(2) Let O be the centre of gravity;

$p_1\sin\text{A} = p_2\sin\text{B} = p_3\sin\text{C} = \frac{2}{3}\text{R}\sin\text{A}\sin\text{B}\sin\text{C}$, $1/\theta = p_1\text{L}/\sin\text{A}$;

hence, taking $\text{L} = \sin^2\text{A}$, $\text{M} = \sin^2\text{B}$, $\text{N} = \sin^2\text{C}$,

$$1/\theta = \tfrac{2}{3}\text{R}\sin\text{A}\sin\text{B}\sin\text{C};$$

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and the locus of P is a circle if

$$\frac{PE \cdot PF}{OE \cdot OF}\sin^2 A + \frac{PF \cdot PD}{OF \cdot OD}\sin^2 B + \frac{PD \cdot PE}{OD \cdot OE}\sin^2 C = K,$$

the radius being given by $\rho^2 = R^2(1 + \frac{1}{2}K \sin A \sin B \sin C)$.

(3) Let O be the orthocentre;

$p_1 \cos A = p_2 \cos B = p_3 \cos C = 2R \cos A \cos B \cos C$, $1/\theta = p_1 L/\sin A$; hence, taking $L = \sin 2A$, $1/\theta = 4R \cos A \cos B \cos C$.

The locus of P is a circle if

$$\frac{PE \cdot PF}{OE \cdot OF}\sin 2A + \frac{PF \cdot PD}{OF \cdot OD}\sin 2B + \frac{PD \cdot PE}{OD \cdot OE}\sin 2C = K,$$

and its radius is given by $\rho^2 = R^2(1 + 2K \cos A \cos B \cos C)$.

(4) Let O be the centre of the circumcircle;

$$\frac{p_1}{\cos A} = \frac{p_2}{\cos B} = \frac{p_3}{\cos C} = R, \quad \frac{1}{\theta} = \frac{p_1 L}{\sin A} = \frac{R \cos A}{\sin A} L;$$

hence, taking $L = \tan A$, $M = \tan B$, $N = \tan C$, $1/\theta = R$, the locus of P is a circle if $\frac{PE \cdot PF}{OE \cdot OF}\tan A + \frac{PF \cdot PD}{OF \cdot OD}\tan B + \frac{PD \cdot PE}{OD \cdot OE}\tan C = K$, and its radius is given by $\rho^2 = R^2(1 + \cos A \cos B \cos C)$.

14318. (H. MacColl, B.A.)—Two intersecting circles A and B are wholly enclosed in a circle E. Six points are marked in A outside B, three points in B outside A, two points in area common to A and B, and x points in E outside both A and B. Out of the total $x + 11$ points in E a point P is taken at random. Let α denote the statement that P will be in A, and β the statement that P will be in B. What must be the value of x (1) so that α shall be independent of β, (2) so that the dependence of α upon β shall be 2/105, (3) so that the dependence of α upon β shall be $-2/95$? [The *dependence of* α *upon* β means the increase or (when negative) diminution produced in the chance of α by assuming β.]

Solution by H. W. Curjel, M.A.

If α and β are independent, $\frac{3}{5} = 8/(x + 11)$; therefore $x = 9$.

If dependence of α upon $\beta = \frac{2}{105}$, $\frac{3}{5} - 8/(x + 11) = \frac{2}{105}$; therefore $x = 10$.

If dependence of α upon $\beta = -\frac{2}{95}$, $\frac{3}{5} - 8/(x + 11) = -\frac{2}{95}$; therefore $x = 8$.

14246. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely $(1440^{10} + 1)$.

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Solution by the PROPOSER.

Let $N = x^{10}+y^{10} = (x^2+y^2).\dfrac{x^{10}+y^{10}}{x^2+y^2} = N_2.N_{10}$, suppose.

Now, by a result ascribed to M. AURIFEUILLE (see ED. LUCAS's *Sur la Série récurrente de Fermat*, Rome, 1879, p. 7).

$$N_{10} = (x^{10}+y^{10})+(x^2+y^2) = L.M,$$

where $L = P-\sqrt{(10xy)}\,Q$, $M = P+\sqrt{(10xy)}\,Q$,

and $P = x^4+5x^3y+7x^2y^2+5xy^3+y^4$, $Q = x^3+2x^2y+2xy^2+y^3$.

Hence N_{10} is always resolvable into the two factors L, M when $10xy = \square$.

Now let $x = 1$, $y = 1440$, $\sqrt{(10xy)} = 120$, giving

$N_2 = 2,073,601$, a *prime* (by BURCKHARDT's *Tables*);

$L = 3,955,945,312,681$; $M = 4,673,577,492,121$.

Now REUSCHLE's *Tafeln complexer Primzahlen*, Berlin, 1875, p. 471, gives 41, 61, 241, 401, 941 as divisors of N_{10}, and shows that N_{10} has no more divisors less than 1000. Also a MS. table prepared by the PROPOSER (and which it is hoped to publish shortly) of solutions of the congruence $(x^{10}+1)\div(x^2+1) \equiv 0 \pmod{p}$, for all primes less than 10,000, shows that 61×61 and 1061 are divisors of N_{10}, and that N_{10} has no more divisors less than 10,000. By division, it is now found that

$$L = 41.241.401.941.1061; \quad M = 61.61.1,256,000,401.$$

The large (10-figure) factor in M is found to be *prime* by LAWRENCE's method given in *Proc. Lond. Math. Soc.*, Vol. XXVIII., "On the Determination of Certain Primes."

And, *finally*, $N = N_2.N_{10} = N_2.L.M$.

14323. (V. DANIEL.)—If the images of the vertices A, B, C of a triangle (area = Δ), in the opposite sides, form a triangle of area Δ′, show that $\Delta'/\Delta = 3+8\cos A\cos B\cos C$. Hence show that, when the images are collinear, the triangle ABC has an obtuse angle lying between the limits 120° and $\cos^{-1}(-\sqrt{\frac{3}{8}})$, *i.e.* nearly 127° 46′.

Solution by the PROPOSER.

AA′, BB′, CC′ meet in O, the orthocentre of the triangle ABC.

Let $2p_A$, $2p_B$, $2p_C$ denote AA′, BB′, CC′; let 2α, 2β, 2γ denote OA, OB, OC. Then

$$\begin{aligned}\Delta' &= \tfrac{1}{2}\sum_{A,B,C,A',B',C'} OA.OB'.\sin C = 2\sum_{A,B,C,\alpha,\beta,\gamma}(p_A-\alpha)(p_B-\beta)\sin C\\ &= 2\left\{\Sigma p_A p_B\sin C+\Sigma\alpha\beta\sin C-\Sigma p_A(\beta\sin C+\gamma\sin B)\right\}\\ &= 2\Sigma a.p_A\sin^2 C+\tfrac{1}{2}\Sigma OA.OB.\sin C-\Sigma a.p_A\\ &= 4\Delta\Sigma\sin^2 C+\Delta-6\Delta.\end{aligned}$$

Therefore $\Delta'/\Delta = 4\Sigma\sin^2 C-5 = 3+8\cos A\cos B\cos C$.

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When the images are collinear, Δ' vanishes, giving

$$3 + 8 \cos A \cos B \cos C = 0,$$

and therefore angle C, say, obtuse, *i.e.*,

$$3 + 4 \cos C \{\cos (A \sim B) - \cos C\} = 0;$$

therefore $$\cos (A \sim B) = \cos C - \tfrac{3}{4} \sec C$$

and $$\{d \cos (A \sim B)\} / (d \cos C) = 1 + \tfrac{3}{4} \sec^2 C = \text{positive quantity.}$$

Hence $(A \sim B)$ and C increase and diminish together, since they are respectively angles in the first and second quadrants. $(A \sim B) = 0$ and $(A \sim B) = \pi - C$ give the minimum and maximum values of C, on substituting in the above quadratic in cos C.

14282. (Professor A. Droz-Farny.)—Dans le triangle ABC de forme donnée, tan A = 1, tan B = 2, tan C = 3; démontrer que la droite d'Euler est parallèle au côté AC.

Solution by Professor Sanjána, M.A.; *and* H. W. Curjel, M.A.

Let G be the centroid, GK perpendicular to AC; let BE, perpendicular to AC, contain the orthocentre O. Then $GK = \frac{1}{3}BE = \frac{2}{3}R \sin A \sin C$, and $OE = 2R \cos A \cos C$; hence $GK/OE = \frac{1}{3} \tan A \tan C = 1$, so that GO is parallel to AC. [It is enough if tan A, tan B, tan C are in *any* arithmetic progression.]

14280. (H. A. Webb.)—Show that a particle dropped from rest at any height will, relatively to an observer on the earth, describe a curve on the surface of a paraboloid of revolution whose axis is parallel to the axis of the earth; it being assumed that the earth rotates on its axis uniformly and exerts a constant attraction on the particle.

Solution by the Proposer.

Take the point of projection for origin, the upward vertical for z-axis, the direction of North for x-axis, and the direction of West for y-axis. If ω be the angular velocity of the rotation of the earth, the exact equations of motion, the earth's centre being considered fixed, are (as in Besant's *Dynamics*, chap. x.)

$$\ddot{x} - 2\dot{y}\omega l + l\omega^2 (zm - xl) = 0, \quad \ddot{y} - 2\omega (\dot{z}m - \dot{x}l) - \omega^2 y = 0 \quad \ldots (1, 2),$$

$$\ddot{z} + 2\dot{y}\omega m - \omega^2 m (zm - xl) = -g \quad \ldots\ldots\ldots (3),$$

where $\sin^{-1} l = \cos^{-1} m =$ the latitude of the place.

Multiply (3) by m, (1) by l, and subtract. Therefore

$$(D^2 - \omega^2)(zm - xl) + 2\omega D y = -mg.$$

From (2) $$(D^2 - \omega^2) y - 2\omega D (zm - xl) = 0,$$

whence $$(D^2 + \omega^2)^2 (zm - xl) = \omega^2 mg, \quad (D^2 + \omega^2)^2 y = 0 \quad \ldots\ldots \text{(i., ii.)}.$$

Multiply (1) by m, (3) by l, and add. Therefore

$$m\ddot{x} + l\ddot{z} = -lg \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii.)}.$$

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We thus have three integrals

$$\left.\begin{aligned} zm - xl &= (A + Bt)\cos\omega t + (A' + B't)\sin\omega t + mg/\omega^2 \\ y &= (C + Dt)\cos\omega t + (C' + D't)\sin\omega t \\ mx + lz &= A'' + B''t - \tfrac{1}{2}lgt^2 \end{aligned}\right\}.$$

The problem of finding the subsequent motion of a particle projected in any manner whatever, at a distance from the earth not comparable with the earth's radius, is therefore completely solved.

Next, suppose the particle dropped from rest at a height h. Initially

$$t = x = y = z = \dot{x} = \dot{y} = \dot{z} = 0.$$

Hence

$$\left.\begin{aligned} x &= lm\,(h - g/\omega^2)(2\sin^2\tfrac{1}{2}\omega t - \omega t\sin\omega t) - \tfrac{1}{2}lmgt^2 \\ y &= m\,(h - g/\omega^2)(\sin\omega t - \omega t\cos\omega t) \\ z &= -m^2\,(h - g/\omega^2)(2\sin^2\tfrac{1}{2}\omega t - \omega t\sin\omega t) - \tfrac{1}{4}l^2gt^2 \end{aligned}\right\}.$$

We can obtain one equation which is independent of t and does not contain transcendental functions. It is

$$y^2 + \left[lx - mz - m\,(h - g/\omega^2)\right]^2 = m^2\,(h - g/\omega^2)^2\left[1 - 2\omega^2/lg\,(mx + lz)\right] \ldots \text{(iv.)}.$$

This is the equation of a paraboloid of revolution whose axis is parallel to the axis of the earth.

14259. (Professor Neuberg.)—Eliminer x, y, z entre les équations

$$(x + y)(x + z) = ayz, \quad (y + x)(y + z) = bzx, \quad (z + x)(z + y) = cxy.$$

Solutions (i.) *by* H. J. Woodall, A.R.C.S.; H. W. Curjel, M.A.; *and others*; (ii.) *by* F. E. Cave; *and* Kela Deva Rau.

(i.) If we multiply (1) by (2), divide the result by (3), and take the square root, we get

$$(x + y + z)/z = (\sqrt{ab} + \sqrt{c})/\sqrt{c} = (\sqrt{abc} + c)/c.$$

Similarly $(x + y + z)/x = (\sqrt{abc} + a)/a$ and $(x + y + z)/y = (\sqrt{abc} + b)/b$.

By inverting these results, and then adding, we get the required eliminant

$$1 = \frac{a}{\sqrt{abc} + a} + \frac{b}{\sqrt{abc} + b} + \frac{c}{\sqrt{abc} + c}.$$

Further work gives $\{abc - (bc + ca + ab)\}^2 = 4abc.$

(ii.) $$(x + y)(x + z) = ayz, \quad (y + z)(y + x) = bxz;$$

therefore

$$z = \frac{x\,(x + y)}{ay - (x + y)} = \frac{y\,(x + y)}{bx - (x + y)};$$

therefore

$$bx^2 - ay^2 = x^2 - y^2$$

or

$$x^2/(a - 1) = y^2/(b - 1) = z^2/(c - 1),$$

by symmetry.

The third equation now gives

$$\left\{\sqrt{(c-1)} + \sqrt{(a-1)}\right\}\left\{\sqrt{(c-1)} + \sqrt{(b-1)}\right\} = c\sqrt{(a-1)}\sqrt{(b-1)}$$

$$\sqrt{(c-1)}\left\{\sqrt{(a-1)} + \sqrt{(c-1)} + \sqrt{(b-1)}\right\} = (c-1)\sqrt{(a-1)}\sqrt{(b-1)}.$$

Dividing by $\sqrt{(c-1)}$ we get the final result

$$\sqrt{(a-1)} + \sqrt{(b-1)} + \sqrt{(c-1)} = \sqrt{(a-1)}\sqrt{(b-1)}\sqrt{(c-1)}.$$

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13969. (C. WAFELBAKKER.)—The triangle ABC is supposed to revolve round the axis BC, till the vertex A, having described a semicircle, has reached the position A′. In the same manner the points B′ and C′ are derived from B and C. It is asked how to construct ABC, being given the points A′, B′, C′.

Solution by Professor F. MORLEY, Sc.D.

A convenient way of applying analysis to many geometric questions is to use circular coordinates, that is, to write $r = X + iY$, $y = X - iY$, where X, Y are rectangular coordinates. The circular coordinates of three points may be taken as α, $1/\alpha$; β, $1/\beta$; γ, $1/\gamma$, the points being taken on the circle $xy = 1$. The join of β and γ is $x + y\beta\gamma = \beta + \gamma$; the perpendicular on this line from α is $x - y\beta\gamma = \alpha - \beta\gamma/\alpha$, and for the intersection $2x_1 = \alpha + \beta + \gamma - \beta\gamma/\alpha$. If, then, a, b, c be the reflexions of α, β, γ in the opposite sides

$$\alpha + a = 2x_1 \quad \text{and} \quad a = \beta + \gamma - \beta\gamma/\alpha \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To solve the three equations of the type (1), we have

$$\alpha - b = \gamma(\alpha - \beta)/\beta, \quad \alpha - c = \beta(\alpha - \gamma)/\gamma,$$

$$(\alpha - b)(\alpha - c) = (\alpha - \beta)(\alpha - \gamma) = \alpha(\alpha - a);$$

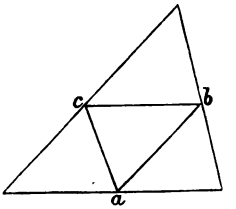
therefore

$$a = bc/(b + c - a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Noting that $b + c - a$ is the fourth point of a parallelogram whose other points are a, b, c, we see that a, β, γ are constructed at once from (2) by similar triangles, *e.g.*, from the fact that a, 0, b is similar to c, 0, $b + c - a$, when once the circumcentre of a, β, γ, which we have assumed as origin is known. But we have, from (2),

$$| bc/(b + c - a) | = \text{circumradius},$$

and therefore



$$| a(b + c - a) | = | b(c + a - b) | = | c(a + b - c) |.$$

The point sought is, then, an intersection of two circular cubics.

Taking as origin the centroid of a, b, c, the equation of a cubic is

$$| (x - b)(x + 2b) | = | (x - c)(x + 2c) |,$$

or, denoting the conjugates of b, c by $\bar{b}$, $\bar{c}$,

$$(x^2 + bx - 2b^2)(y^2 + \bar{b}y - 2\bar{b}^2) = (x^2 + cx - 2c^2)(y^2 + \bar{c}y - 2\bar{c}^2) \ldots\ldots (3).$$

The tangents at the circular points meet where

$$x^2 + bx - 2b^2 = x^2 + cx - 2c^2,$$

i.e., at

$$x = 2(b + c) \quad \text{or} \quad x = -2a.$$

This point is on the curve, which is thus a "focale."

A focale may be regarded as the locus of the feet of normals from a

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point (the double focus) to a system of coaxial circles. We proceed to find the base points (K, K′) of this system.

Denote the triangle $-2a-2b-2c$ by ABC; the triangle *abc* by A′B′C′. The cubic is defined by

$$PB \cdot PB' = PC \cdot PC'.$$

The point for which PB = PC, PB′ = PC′ is at infinity in a direction perpendicular to BC. The line through A in this direction meets the curve again at a point on the parallel to BC through the centroid G; this is shown by a simple calculation. Thus the base-points lie on this parallel.

Again, the point P_1 for which PB = PC′, PC = PB′ divides AO, where O is the circumcentre of ABC, in the ratio -3. Calculating the third intersection of AO and the curve, we find that it is A itself. Therefore the base-points are on the circle AP_1.

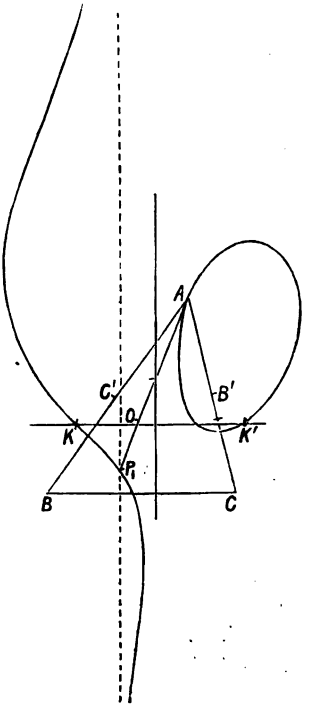
When the base-points are real, the curve is bipartite; when imaginary, unipartite. In our case the test is whether

$$| \tfrac{3}{4}p - \tfrac{3}{4}R\cos(B-C) | < \tfrac{3}{4}R,$$

where p is the distance from A to BC, R the circumradius; or whether

$$| \cos(B-C) + 8\cos(B+C) | < 9,$$

which is true. Thus *the cubics are bipartite.*



For a mechanical construction of the curve we may place the corners of slotted T-squares at the base-points, and place pencils at a pair of their intersections. These pencils must then be held in line with a fixed point A by a slotted bar.

Or, fix one point of a jointed rhombus KLMN at K; let L move on a fixed line, while LM passes through a fixed point A. Then M will describe the curve.

These constructions apply only to the bipartite curves; but we have shown that those alone are wanted.

[Prof. G. B. Mathews, F.R.S., makes the following remarks:—

$$a'^2 = b^2 + c^2 - 2bc\cos 3A = b^2 + c^2 - (b^2 + c^2 - a^2)\left\{\frac{(b^2+c^2-a^2)^2}{b^2c^2} - 3\right\}.$$

Putting $x, y, z, p, q, r,$

for $a^2, b^2, c^2, a'^2, b'^2, c'^2,$

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we have $pyz = yz(y+z)-(y+z-x)^3+3yz(y+z-x)$

$= 4yz(y+z)-(y+z-x)^3-3xyz$

$= x^3-3(y+z)x^2+3(y^2+yz+z^2)x-(y+z)(y-z)^2$ .....(1).

Similarly $qzx = 4zx(z+x)-(z+x-y)^3-3xyz$

$= y^3-3(z+x)y^2+3(z^2+zx+x^2)y-(z+x)(z-x)^2$ .....(2),

$rxy = 4xy(x+y)-(x+y-z)^3-3xyz$

$= z^3-3(x+y)z^2+3(x^2+xy+y^2)z-(x+y)(x-y)^2$ .....(3).

From these equations x, y, z must be found in terms of p, q, r. Suppose $y = \lambda x$, $z = \mu x$; then

$$p\lambda\{4\mu(\mu+1)-(\mu+1-\lambda)^3-3\lambda\mu\} - q\{4\lambda\mu(\lambda+\mu)-(\lambda+\mu-1)^3-3\lambda\mu\} = 0 \quad\ldots\ldots(4),$$

$$p\mu\{4\lambda(\lambda+1)-(\lambda+1-\mu)^3-3\lambda\mu\} - r\{4\lambda\mu(\lambda+\mu)-(\lambda+\mu-1)^3-3\lambda\mu\} = 0 \quad\ldots\ldots(5).$$

Equations (4), (5) are satisfied by $\lambda = 0$, $\mu = 1$, $\lambda = 1$, $\mu = 0$, and also by $\lambda = \mu = \infty$. The elimination of either λ or μ is *possible*, although very laborious. I cannot see *a priori* the degree of the resulting reduced equation; I doubt whether it can in any case be solved by quadratics; if not, then there is no geometrical solution by rule and compass.

Considering λ, μ as current coordinates, (4) and (5) represent two quartics, with 16 intersections. Now

$\lambda = 0$, $(\mu-1)^3 = 0$, gives 3,

$\mu = 0$, $(\lambda-1)^3 = 0$, ,, 3 more,

$\lambda-\mu = 0$, $\lambda = \infty$, ,, 3 ,, (at infinity).

There remain 7 to be accounted for; the equation in λ must be of the form $\lambda^3(\lambda-1)^3 f(\lambda) = 0$, where f is of degree 7 and with coefficients rational in p, q, r. Now it seems clear that we cannot have $\lambda =$ a rational function of p, q, r; hence *either* there is at least one other purely numerical value (this I doubt), or else $f(\lambda)$ is irreducible, and this would account for the ill-success of attempts to solve the problem.

P.S.—The remarks as to the degree and irreducibility of $f(\lambda)$ are, of course, merely conjectural; but the correctness of the guess is to some extent confirmed by Professor Morley's solution. For, if (as appears to be the case) his two cubics *cut* at the circular points and have no other intersections at infinity, there will be a set of seven finite intersections which, in all probability, are algebraically inseparable in the general case. It would be interesting to actually work out $f(\lambda)$, in order to see what relations connecting p, q, r suffice to make the equation $f(\lambda) = 0$ reducible or solvable by radicals. (*Cf.* Weber's *Algebra*, II., p. 530.)

Mr. H. A. Webb gives the following interesting analysis of the Question:—If a, b, c be the sides of the triangle ABC, and l, m, n the sides of the triangle A'B'C', and if, for brevity, we write

$$\alpha = 2(m^2-n^2), \quad \beta = 2(m^2+n^2-l^2), \quad \lambda = l^2,$$

then $(2a^2-l^2)$ is a root of the equation in y,

$$64y^7-48\beta y^6-(124\lambda^2+52\beta\lambda-9\beta^2+13\alpha^2)y^5 - \left[68\lambda^3+12\lambda^2\beta-\lambda(7\beta^2+13\alpha^2)-8\alpha^2\beta\right]y^4+(44\alpha^2\lambda^2+14\alpha^2\beta\lambda-\alpha^4)y^3 + (20\alpha^2\lambda^2+6\alpha^2\beta\lambda-3\alpha^4)\lambda y^2-3\alpha^4\lambda^2 y-\alpha^4\lambda^3 = 0.$$

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This result is interesting only in so far as it goes to prove the improbability of the existence of a purely geometrical solution. But, from a strictly analytical point of view, it can be regarded as a complete solution of the problem, since when the sides of the triangle ABC are known the triangle can at once be constructed.

For some further remarks on this Question see p. 29 of the present Vol.]

14234. (D. Biddle.)—As the angle A of a variable triangle increases from 0 to π, the sides AB and AC increase in the same proportion with it. Supposing the dimensions at some period of the transformation to be given, and AB to remain fixed in direction, find (1) the maximum height of C above AB, (2) the maximum area attained, (3) the locus of the centroid.

Solution by the Proposer.

Let b, c = AC, AB when A is a right angle.

(1) Here $x \sin x$ is a maximum. Therefore $x \cos x + \sin x = 0$, and $\tan x = -x$. Consequently the height of C is a maximum when the arc of A equals the tangent of the supplementary angle. This is the case approximately when A = 116° 14½′, the arc of which is 2·0287513, and equals tan 63° 45½′ nearly. The height of C is given by

$$\sin A\,(2{\cdot}0287513/1{\cdot}5707963)\,b \quad \text{or} \quad 1{\cdot}15846b \text{ nearly.}$$

(2) Here $x^2 \sin x$ is a maximum; whence $x^2 \cos x + 2x \sin x = 0$, and $\tan x = -\frac{1}{2}x$. This is the case approximately when A = 131° 8½′, its arc being 2·2889266, whilst tan 48° 51½ = 1·1444633 nearly. Hence the maximum area of the variable triangle is ·799586bc, bearing to the area when A is a right angle, the ratio 8 : 5 nearly. [When, as in (1), C is at its maximum height, Δ = ·7481bc.]

(3) Let ρ equal the radius vector, measured from A, and θ equal the angle formed with AB. Then, if the parallelogram BACD be completed, AD = 3ρ. Again let ϕ = ADB; then A = $\theta + \phi$, and $\sin\phi = (c/b)\sin\theta$, whilst $\cos\phi = (b^2 - c^2\sin^2\theta)^{\frac{1}{2}}/b$. Now

$$3\rho\sin\theta = \mathrm{AC}\sin\mathrm{A} = \frac{2\mathrm{A}}{\pi}b\sin\mathrm{A} = \frac{2b}{\pi}(\theta+\phi)\sin(\theta+\phi)$$

$$= \frac{2b}{\pi}\left\{\theta + \sin^{-1}(c/b)\sin\theta\right\}\left\{\frac{c\sin\theta\,.\cos\theta + (b^2 - c^2\sin^2\theta)^{\frac{1}{2}}\sin\theta}{b}\right\}.$$

$$\therefore\quad \rho = \frac{2}{3\pi}\left\{\theta + \sin^{-1}(c/b)\sin\theta\right\}\left\{c\cos\theta + (b^2 - c^2\sin^2\theta)^{\frac{1}{2}}\right\}.$$

When $b = c = 1$ this reduces to $\rho = 8\theta\,.\cos\theta/(3\pi)$, which is a particular form of $\rho = ax\cos x$. And it is interesting to note that in every case

$$\sin\theta : \cos\theta = \sin\mathrm{A} : \cos\mathrm{A} + c/b.$$

The locus of C is the Spiral of Archimedes (invented by Conon, B.C. 250), for here $\rho = (2b/\pi)\theta$.

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14194. (Professor FRANZ ROGEL.)—Prove that

$$S_r = q + \sum_{\kappa=3,5,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\kappa-1)}}{\kappa!} (2\pi m)^{\kappa-1} \sum_{\nu=1,2,\ldots}^{q} B_\kappa(\nu)\, \nu^{r-\kappa} \quad \ldots\ldots (1),$$

where $S_r = \Sigma t^r$, t divisor of $m \leqq q$, and B_κ denotes the "Bernoullian function" of the κth order.

$$E\left(\frac{m}{n}\right) = m + \frac{1}{2\pi} \sum_{\kappa=3,5,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\kappa-1)}}{\kappa \cdot \kappa!} \left(\frac{2\pi}{n}\right)^{\kappa} B_\kappa(m+1)\, B_\kappa(n) \ldots (2),$$

where $E\left(\frac{m}{n}\right)$ represents the greatest integer $\leqq \frac{m}{n}$.

Solution by the PROPOSER.

It is supposed that m and n are integers, and $\theta = (m/n)\pi$,

$$\frac{\sin 2m\pi}{2\tan(m/n)\pi} = \frac{\sin 2n\theta}{2\tan\theta} = \frac{\sin(2n-1)\theta + \cos 2n\theta \cdot \sin\theta}{2\sin\theta}$$

$$= \frac{\sin(2n-1)\theta}{2\sin\theta} + \tfrac{1}{2}\cos 2n\theta = 1 + \cos 2\theta + \cos 4\theta \ldots + \cos 2(n-1)\theta$$

$$= 1 + \sum_{\sigma=0,2,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}\sigma}}{\sigma!} (2\theta)^\sigma \left\{1^\sigma + 2^\sigma \ldots + (n-1)^\sigma\right\},$$

where $$1^\sigma + 2^\sigma \ldots + (n-1)^\sigma = \begin{cases} 1/(\sigma+1)\, B_{\sigma+1}(n), & \sigma > 0, \\ B_1(n), & \sigma = 0, \end{cases}$$

$B_{\sigma+1}(n)$ being the "Bernoullian function" of the $(\sigma+1)$th order.

Therefore $$\frac{\sin 2m\pi}{2\tan\theta} = \sum_{\sigma=0,2,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}\sigma}}{(\sigma+1)!} (2\theta)^\sigma B_{\sigma+1}(n),$$

or, taking λ for $\sigma+1$,

$$\frac{\sin 2m\pi}{2n\tan(m/n)\pi} = \frac{1}{2m\pi} \sum_{\lambda=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!} (2m\pi/n)^\lambda B_\lambda(n) \quad \ldots\ldots (\alpha)$$

$$\equiv (m, n) = \begin{cases} 1, & m/n \text{ an integer}, \\ 0, & m/n \text{ a fraction}, \end{cases}$$

m and n integers.

In order to prove the formula (1), we have $S_r \equiv \Sigma t^r$, where t denotes the divisors of m, which are $\leqq q$; therefore

$$S_r = \sum_{\nu=1,2,\ldots}^{q} (m, \nu)\, \nu^r,$$

and, in consequence of (α),

$$S_r = \frac{1}{2m\pi} \sum_{\lambda=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!} (2m\pi)^\lambda \sum_{\nu=1,2,\ldots}^{q} B_\lambda(\nu)\, \nu^{r-\lambda} \ldots\ldots (1).$$

Hence, for $r = 0$, $q = m$, the *number of divisors* of m

$$S_0 = \frac{1}{2m\pi} \sum_{\lambda=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!} (2m\pi)^\lambda \sum_{\nu=1,2,\ldots}^{m} \frac{B_\lambda(\nu)}{\nu^\lambda} \quad \ldots\ldots (1^*),$$

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where

$$\frac{B_\lambda(\nu)}{\nu^\lambda} = 1 - \frac{\lambda}{2\nu} + \binom{\lambda}{2}\frac{B_1}{\nu^2} - \binom{\lambda}{4}\frac{B_2}{\nu^4} \ldots + (-1)^{\frac{1}{2}\lambda+1}\binom{\lambda}{\lambda-1}\frac{B_{\frac{1}{2}(\lambda-1)}}{\nu^{\lambda-1}}.$$

Then we have for the *greatest integer*

$$E\left(\frac{m}{n}\right) = \sum_{\mu=1}^{m}(\mu, n),$$

and, according to (α),

$$E\left(\frac{m}{n}\right) = \frac{1}{2\pi}\sum_{\lambda=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!}\left(\frac{2\pi}{n}\right)^\lambda B_\lambda(n)\sum_{\mu=1}^{m}\mu^{\lambda-1}$$

$$= -1 + \frac{1}{2\pi}\sum_{\lambda=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!\,\lambda}\left(\frac{2\pi}{n}\right)^\lambda B_\lambda(n)\,B_\lambda(m+1) \quad \ldots (2).$$

Respecting $\sum_{\mu=1}^{m}\mu^0 = B_1(m+1) - 1$;

by taking km, kn for m, n, where k is an integer, we obtain a more general formula

$$E\left(\frac{m}{n}\right) \equiv E\left(\frac{km}{kn}\right) = -1 + \frac{1}{2\pi}\sum_{\lambda=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!\,\lambda}\left(\frac{2\pi}{kn}\right)^\lambda B_\lambda(kn)\,B_\lambda(km+1)$$

..... (2*).

Or, respecting $E\left(\frac{m}{n}\right) = \sum_{\mu=1}^{m}(k\mu, kn)$,

we find

$$E\left(\frac{m}{n}\right) = -1 + \frac{1}{2\pi k}\sum_{\lambda=1,3,5\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(\lambda-1)}}{\lambda!\,\lambda}\left(\frac{2\pi}{n}\right)^\lambda B_\lambda(m+1)\,B_\lambda(kn) \ldots (2^{**}),$$

k, m, n integers.

14052. (B. N. Cama, M.A.)—Show that the sum of all numbers greater than 10^{n-1} which can be formed with the n digits $a_1, a_2, a_3, \ldots, a_n$ (no digit being repeated in the same number) is $\frac{1}{9}S(10^n-1)(n-1)!$, S representing the sum of the n digits. Find the corresponding sum when one of the digits (a_n, say) vanishes.

Solutions (1) *by* Lt.-Col. Allan Cunningham, R.E.; (2) *by* L. E. Reay, B.A.

(1) Let N denote any of the numbers; r the radix of scale of notation.

$$N = a_1 + a_2 r + a_3 r^2 + \ldots + a_n r^{n-1}.$$

And, because every $N > r^{n-1}$, therefore every N contains *all* the n digits (no two digits alike, and no zero). Let P_n denote the total number of numbers N that can be formed by permuting the n digits; it may be easily shown that $P_n = n!$. Hence, omitting *one* digit (say a_n), P_{n-1} numbers of $(n-1)$ digits each can be formed, and $P_{n-1} = (n-1)!$. Hence also P_{n-1} numbers of n digits each can be formed with the particular

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digit a_n in *any* given position (say the xth place). Similarly, in the whole number (P_n) of numbers of n digits, *every* digit ($a_1, a_2, \ldots, a_n$) occupies a given place the *same number of times*, viz., P_{n-1} times. Hence, in forming the sum of the numbers N, every power of r in the expression for N becomes multiplied by $S = (a_1 + a_2 + \ldots + a_n)$, and also by P_{n-1} = the number of times *each* digit multiplies that power of r. Therefore

$$\Sigma_n(N) = S(1 + r + r^2 + \ldots + r^{n-1})P_{n-1} = S\frac{r^n - 1}{r - 1}(n-1)!,$$

as required.

Secondly, if one of the digits be zero, it cannot occupy the highest place (multiplying r^{n-1}), since every $N > r^{n-1}$. Now the sum of all the numbers $> r^{n-2}$ that could be formed with the remaining $(x-1)$ digits is (the sum of the digits being the same as before)

$$\Sigma_{n-1}(N) = S(1 + r + r^2 + \ldots + r^{n-2})P_{n-2} = S\frac{r^{n-1} - 1}{r - 1}(n-2)!.$$

This sum must be *subtracted from the former*.

(2) Numbers $> 10^{n-1}$; therefore all digits must be used. There are $(n-1)!$ numbers with a_n or a_{n-1}, &c., as units digit. Therefore

sum of units digits $= (n-1)!\,S$.

Similarly, sum of tens digits $= (n-1)!\,S \cdot 10$, and so on.

Therefore sum of numbers $= (n-1)!\,S(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)$

$= \frac{1}{9}(n-1)!\,S(10^n - 1)$.

If a_n is zero, it can never be in the 10^{n-1} place; therefore there are $(n-2)(n-2)!$ numbers with a_1, a_2, &c., in units place, and so on as above to 10^{n-2} place. Therefore

sum of digits to 10^{n-2} place $= (n-2)(n-2)!\,S(10^{n-2} + \ldots + 10 + 1)$.

With a_1 in 10^{n-1} place, $(n-1)!$ numbers can be formed. Therefore

sum of 10^{n-1} digits $= (n-1)!\,S\,10^{n-1}$.

Therefore sum of numbers $= (n-2)!\,S\{\frac{1}{9}(n-2)(10^n - 1) + (n-1)\,10^{n-1}\}$

$= \frac{1}{9}(n-2)!\,S\{(10n-11)\,10^{n-1} - (n-2)\}$.

E.g.— $a_1 = 0$, $a_2 = 1$, $a_3 = 2$.

Therefore $S = 3$, $n = 3$.

Sum $= \frac{1}{9} \cdot 3(1900 - 1)$

$= \frac{1}{3}(1899)$

$= 633$.

Numbers.	
210	
201	
120	
102	
Total	633

If l digits vanish.—Take one of non-vanishing digits in units place; $\frac{(n-2)!}{l!}(n-l-1)$ numbers can be found. Therefore, as before,

sum to the 10^{n-2} place $= (n-l-1)\frac{(n-2)!}{l!}\left(\frac{10^{n-1} - 1}{9}\right)S$,

sum of digits in 10^{n-1} place $= \frac{S(n-1)!}{l!}10^{n-}$

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Therefore sum $= \frac{S(n-2)!}{l!}\left\{\frac{(n-l-1)(10^{n-1}-1)}{9} + (n-1)\,10^{n-1}\right\}$.

E.g.—1, 2, 0, 0.

$n = 4, \quad l = 2, \quad S = 3.$

By formula,

$$\text{sum} = \frac{3(2!)}{2!}\left(\frac{10^3-1}{9} + 3.10^3\right)$$

$$= 3(3111)$$

$$= 9333.$$

Numbers.	
2100	
2010	
2001	
1200	
1020	
1002	
Total	9333

Again, if $n-l > 9$, some of the non-vanishing digits are repeated. The formula would doubtless be found by a similar method.

[Mr. Curjel solves the Question as follows:—"All the digits must be used; therefore a_p occurs $(n-1)!$ times in each place; therefore

$$\text{sum} = S\frac{r^n-1}{r-1}(n-1)! = \tfrac{1}{9}S(10^n-1)(n-1)! \text{ if } r = 10.$$

If one of the digits is zero, we must subtract the sum of all the numbers which begin with a zero; *i.e.*, $S\frac{r^{n-1}-1}{r-1}(n-2)!$.

Hence we get $S/(r-1)(n-2)!\left\{r^n(n-1) - r^{n-1} - (n-2)\right\}$

$$= \tfrac{1}{9}S(n-2)!\left\{(n-1)\,10^n - 10^{n-1} - n + 2\right\},$$

if $r = 10$."]

14273. (Rev. T. Mitchson, B.A.)—The angle between a semi-diameter of an ellipse and the tangent at its extremity is a *minimum* when the area of the rectangle contained by the coordinates of the tangential point is a *maximum*, *i.e.*, when $x = a/\sqrt{2}$, $y = b/\sqrt{2}$.

Solution by G. W. Preston *and* H. W. Curjel, M.A.

Let θ be the excentric angle of any point on the ellipse; therefore equation of tangent is $(x/a)\cos\theta + (y/b)\sin\theta = 1$,

and equation of diameter through point is

$$ay\cos\theta - bx\sin\theta = 0.$$

If ϕ be the angle between the diameter and tangent, we have

$$\tan\phi = \frac{(b/a)(\tan\theta + \cot\theta)}{1-(b^2/a^2)},$$

and is therefore a minimum when $\tan\theta + \cot\theta$ is, *i.e.*, when $1/(\sin\theta\cos\theta)$ or $2/\sin 2\theta$ is, *i.e.*, when $2\theta = \frac{1}{2}\pi$ or $\theta = \frac{1}{4}\pi$.

This gives as coordinates of the tangential point $a/\sqrt{2}$, $b/\sqrt{2}$, as required.

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